Try to make sure you can do the (*) probs. ANY could be covered in recitations.(**) indicates a more challenging problem.

1. These are questions about properties of probability measures and about events.

   (a) An experiment has two outcomes, one with probability $1-p$, the other with probability $2p^2$. What is p? Explain your answer.

   (b) TRUE or FALSE: "If $P(A \backslash B) + P(B \backslash C) = P((A \cup B) \backslash C)$ then $B \cap C = \phi$. Decide the issue and explain your answer.

   (c) (*) TRUE or FALSE: "If $P(A) < P(B \backslash A)$ then $P(A) < 1/2$." Decide the issue and explain your answer.

   (d) (*) Prove $P[(A \cap B^c) \cup (A^c \cap B)] = P(A) + P(B) - 2P(A \cap B)$. Describe in English the event whose probability is computed by the expression on the left-hand side of the equation.

2. (S, P) is the probability space of tossing two dice (one red and one blue) under equally likely probability.

   (a) Let $A = \{$red die is $3, 4$, or $5\}$, $B = \{$blue die is 1 or $2\}$, and $C = \{$sum is $7\}$. Are these events pairwise independent?, Are they mutually independent?

   (b) Let $A = \{$red die is odd$\}$, $B = \{$blue die is even$\}$, and $C = \{$the sum is odd$\}$. Compute $P_A(B)$, $P_A(C)$, $P_B(C)$. Are they pairwise independent? Explain. Are they mutually independent? Explain.

   (c) (*) Repeat (b), but now the dice both have TWO faces that say 5 and NO face that says 6.

   (d) (**) Let $A = \{$sum is $7\}$ and $B = \{$there is at least one $6\}$. Compute $P(A)$ and $P_B(A)$. Here is an apparent paradox: compute $P_{C_x}(A)$ where, for $x = 1, 2, 3, 4, 5$, or 6, C_x is the event that there is at least one $x\}$. Since you know that some C_x always occcurs, how can $P(A) \neq P_{C_x}(A)$? Discuss whether this makes sense or not. How do you reconcile it?

3. A computer has printer (P), disk (D) and terminal (T) outputs. Sixty percent of all output characters are on D, thirty percent on P, and the rest on T. The error rate for D is $1/2000$, for P it is $2/1000$, and for T it is $1/1000$. The experiment $\mathcal{E}$ is that a character is output and we observe (i) which type of device made that output and (ii) whether the character was correct. Write down the sample space. What is the probability the character was written on the disk, given $A = \{$it was incorrect$\}$? [Bayes rule].

4. (*) Repeat the previous question but now there is a fourth output device, a magnetic tape (M) with an error rate of $1/500$. And on THIS computer, fifty-five percent of all output characters are on D, twenty-five percent are on P, fifteen percent are on T and five percent on M.

5. A box contains 100 balls. 20 are red, 30 are green, and the rest are yellow. $3/4$ of the red balls are small (the rest are big), $2/3$ of the green balls are small, and $1/2$ of the yellow balls are small. The experiment is to choose a ball at random and to observe its color and its size.

(a) Carefully describe the probability space for this experiment.

(b) What is the probability of the event $A = \{$a small ball is chosen$\}$?

(c) You are told that A occurs. What is $P_A(\text{red})$?

6. (**) There are 4 envelopes, one of which contains 100 dollars, the other 3 being empty. You take one of the envelopes at random.

(a) What is the probability that when you open it, it will contain 100 dollars?

(b) Now, before you open your envelope, somebody opens one of the other 3 and shows you that it is empty. You are now offered the choice to (i) keep your original envelope or (ii) change to one of the remaining 2. What is the probability you win 100 dollars if you do (i)? What is the probability if you do (ii)? Explain in detail.

7. (On independence and nuances).

(a) Decide if A and B can be independent if they are mutually exclusive, and explain your answer.
   • If A and B are mutually exclusive $P(A \cap B) = P(\phi) = 0$ so as long as neither A nor B have probability 0, $P(A)P(B) > 0 = P(A \cap B)$, so they are not independent.

(b) Decide if A and B can be independent if $A \subseteq B$ and explain your answer. You may assume both events have probability larger than zero and smaller than one.
   • NO! We are told that $0 < P(A) = x \leq y = P(B) < 1$ (because $A \subseteq B$). If they were independent, $x = P(A) = P(A \cap B) = P(A)P(B) = xy$, which means y is 0 or 1 (neither allowed).
   Also - intuitively - if $A \subseteq B$, if A occurs, we *know* that B does, implying $P_A(B) = 1$, despite B not being the certain event.

(c) Assume that A, B, C are pairwise independent events. Does it necessarily follow that $A \cup B$ and C are independent? If YES, try to show why. If NO, give an example where it fails.
   • NO (as explained in a class). E.g., take $S = \{1, 2, 3, 4\}$ under equally likely prob., and $A = \{1, 2\}, B = \{1, 3\}, C = \{1, 4\}$, each event with prob $= 1/2$. Clearly $1 = A \cap B = A \cap C = B \cap C$, so they are pairwise independent. But $\{A \cup B\} = \{1, 2, 3\}$ has probability $3/4$, so it fails to obey the product rule with C: i.e., $P((A \cup B) \cap C) = P(\{1\}) = 1/4 \neq P(A \cup B)P(C) = (3/4)(1/2) = 3/8$.

(d) Repeat the above question, now assuming that A, B, C are mutually independent.

(e) (*) Assume A, B, C, D are pairwise independent events. Decide if $A \cap B$ and $B \cap D$ are independent events. Then repeat this assuming the four events are mutually independent.

(f) (**) When are the events $A \cap B$ and C independent? (a) Always; (b) never? (c) sometimes. If you say (c) try to characterize when it occurs.

(g) Describe a probability space (S, P) and events in it (the simpler, the better), where the following hold:
   i. The events A_1, A_2, A_3 are not pairwise independent although A_1 and A_2 are independent and A_1 and A_3 are independent.
   ii. In the family of events $A_1, \ldots, A_4$, the three events A_1, A_2, A_3 are pairwise independent and A_1, A_2, A_4 are pairwise independent, but A_3 and A_4 are not independent.

iii. (**) Events $A_1, \ldots, A_4$ are not 3-wise independent although the events A_1, A_2, A_3, and A_1, A_2, A_4, and A_1, A_3, A_4 are each 3-wise independent.

NOW, SOME COUNTING PROBLEMS:

1. (*) A man is buying a present. He will choose a tie or a shirt. There are 3 ties and 2 shirts to choose from. How many choices has he if he buys one item? How many if he buys one tie and one shirt? What if he buys any two items?

2. How many integers are there larger than one million, smaller than ten million, and having no consecutive digits the same? Explain your counting.

3. (*) A town has 17,910 people. Show that at least two have the same set of 3 initials (everyone has 3 initials, one each for01 the first letter of their Firstname, their Middlename, their Surname).

4. (*) A domino is made by gluing two square pieces together; each square piece has a number from 1 to 9. Order is not important - the domino $\boxed{2\ 3}$ may be used interchangeably with $\boxed{3\ 2}$ in the playing of the game. How many different dominos are there?

5. How many ways can 4 men and 4 women form (heterosexual) couples? How many ways can they stand in a row, alternating sexes? How many ways can they stand in a circle, alternating sexes?

6. (*) Sample twice from $T = \{1, \ldots, 9\}$ without replacement. Find the probability that an odd digit will be selected (a) on the first choice, (b) on the second choice, and (c) on both choices. Explain your answer.

7. How many of the permutations of the first nine positive integers have all the even numbers before any of the odd ones?

   • 4!5! permutations.

8. (*) Use Stirlings's approximation to estimate 30!. The truth is

$$265, 252, 859, 812, 268, 935, 315, 188, 480, 000, 000.$$

   • $\sqrt{2\pi n}(n/e)^n$.

9. How many straight lines can be drawn through six points A, B, C, D, E, F, no three of which are collinear?

   • There are $\binom{6}{2} = 15$ pairs of points, each determining a unique line.

10. (**) Ten basketball players meet for a game. In how many ways can they be divided into two teams of five each? Explain your answer.

    • TRICKY! $\binom{10}{5}/2$. If you dont divide by 2 you count each subset of 5 twice, once when its the chosen 5 and again when its the NOT chosen 5!

11. Use Stirlings's approximation to show that $\binom{2n}{n} \sim 4^n/\sqrt{\pi n}$; i.e., the ratio of the two expressions converges to one.

12. If n balls are randomly placed into n boxes, find the probability that exactly one box is empty. Explain your reasoning.

- $S = \{(b_1, \ldots, b_n), b_i \in \{1, \ldots, n\},$ the box ball i goes in$\}$ and $|S| = n^n$ (every ball could fall into any box). Our event A has $|A| = \binom{n}{2}n[(n-1)_{n-2}] : \binom{n}{2}$ ways to choose the PAIR of balls occupying the same box, n ways to choose which box that would be, and then $[(n-1)(n-2)\cdots(2)]$ ways to place the remaining $n-2$ balls, one each into an empty box.

13. (**) If $n+1$ balls are randomly placed into n boxes, find the probability that exactly one box is empty. Explain your reasoning.

- Similar to the above but more complicated.

  If there were NO empty boxes, ONE of the boxes would have to have two balls and the others would each have one ball. To get from here to ONE empty box choose one of the $n-1$ singly occupied boxes, remove its ball and place it in a different box: if that box is the one box already with two balls, it will now be triply occupied, $n-2$ of the other boxes would have one ball each, and one box would be empty (this is an event T, "T" for a "triply" occupied box); otherwise the ball would join the ball in one of the $n-2$ singly occupied boxes and we now have one empty box, two doubly occupied boxes, and $n-3$ singly occupied boxes (this is the event D, "D" for "Doubles", or a pair of doubly occupied boxes). These observations reveal the <u>structure</u> of the event "one empty box". Now we count.

  The size of the sample space is $|S| = n^{n+1}$ and $|T| = n\binom{n+1}{3}(n-1)_{n-2}$: the first term is for which box will get three balls; the second for which three balls go into that box; the last counts the number of ways the remaining balls can go into the remaining boxes, one ball each (and leaving one empty box).. $|D|$ is $\binom{n}{2}\binom{n+1}{2}\binom{n-1}{2}(n-3)_{n-4}$: the first term is for the number of ways we can choose which two boxes get two balls each; the second and third count the choices of balls for the first doubly occupied box, and then for the second; the last counts the placement of the remaining balls, each into an empty box.