Some Combinatorial and Algorithmic Applications of the Borsuk-Ulam Theorem

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Abstract

The Borsuk-Ulam theorem has many applications in algebraic topology, algebraic geomtry, and combinatorics. Here we study some combinatorial consequences, typically asserting the existence of a certain combinatorial object. An interesting aspect is the computational complexity of algorithms that search for the object. The study of these algorithms is facilitated by direct combinatorial existence proofs that bypass Borsuk-Ulam.

1 Introduction and Summary

The Borsuk-Ulam theorem states that if f is a continuous function from the unit sphere in R^n into R^n, there is a point x where $f(x) = f(-x)$; i.e., some pair of antipodal points has the same image. The recent book of Matoušek [24] is devoted to explaining this theorem, its background, and some of its many consequences in algebraic topology, algebraic geometry, and combinatorics. Borsuk-Ulam is considered a great theorem because it has several different equivalent versions, many different proofs, many extensions and generalizations, and many interesting applications.

A familiar consequence is the ham-sandwich theorem (given d finite continuous measures on R^d, there exist a hyperplane that simultaneously bisects them), along with some of its extensions and generalizations to partitioning continuous measures [2], [6], [7], [8], [10]. In many cases we can derive combinatorial statements that give discrete versions of these results. This, in turn, raises algorithmic issues about the computational complexity of finding the asserted combinatorial object. For example Lo et. al. [23] gave a direct combinatorial proof of the discrete ham-sandwich theorem and described algorithms to compute ham-sandwich cuts for point sets. Various generalizations and extensions were considered in [1], [2], [3], [4], [5], [9], [10], [11], [12], [20], [21], [22], [26], and [30].

A recent interesting example extends a result of Bárány and Matoušek [7], who combined Borsuk-Ulam with equivariant topology to show that three finite, continuous measures on R^2 can be equipartitioned by a *2-fan*, the region spanned by two half-lines incident at a point. Bereg [10] strengthened this statement, proved a discrete version for measures concentrated on a given set of points, and described a beautiful algorithm to find such a partitioning. In Section 2 we show his algorithm to be nearly optimal via a lower bound for this task.

In Section 3 we study several kinds of equitable partitions of a given set $S = \{P_1, \ldots, P_n\}$ of points in general position in R^2. First, there always exists an orthogonal equipartitioning for S; i.e., a pair of orthogonal lines ℓ_1 and ℓ_2 with the property that none of the four quadrants has more than $n/4$ of the points of S. We show that $\Theta(n \log n)$ is the RAM complexity to find such a partitioning, but if the points of S are in convex position, there is a linear time algorithm. Next we study the equitable partitioning guaranteed by Buck and Buck [13], namely the existence of lines ℓ_1, ℓ_2, and ℓ_3 incident at a common point, and having the property that none of the six open regions they define contains more than $n/6$ of the points of S. We show that this partitioning also has complexity $\Theta(n \log n)$, but if the points are in convex position, a linear time solution is possible. Finally we study the interesting "cobweb" equipartitioning discovered by Schulman [27]. This is a convex quadrilateral plus lines ℓ_1 and ℓ_2 through the pairs of opposite vertices and having the property that none of the eight regions defined contains more than $n/8$ points of S. We sketch an $O(n \log n)$ algorithm and conjecture that there is a matching lower bound, except if S consists of points in convex position, where we think the complexity is linear.

In Section 4 we mention some other partitioning problems. Most of these are interesting open questions.

2 Equipartitioning Three Measures by a 2-Fan

A *two-fan* in the plane is a point P (called the center) and two rays, ρ_1 and ρ_2 incident with P. This structure partitions R^2 into two connected regions. Bárány and Matoušek [7] had proved that given three finite, measures on the plane, there is always an equitable partition by a two fan; i.e., there exists a two fan whose two regions have exactly half of each measure. Bereg [10] later considered a discrete version and proved that there are many two fans with equitable partitions of a given input point set. Specifically, given $2r$ red points, $2b$ blue points, and $2g$ green points in general position in R^2, and a line ℓ, there exists a point $P \in \ell$ and a two fan centered at P for which there are r red, b blue, and g green points in both of the regions induced by the fan. He described an $O(n(\log n)^2)$ algorithm to find such a two fan, $n = r + b + g$ being one half the number of points in the problem. Here we show the algorithm to be nearly optimal, by proving

Lemma 1 *Let S be a given set of points in R^2, an of them red, bn of them blue, and cn of them green. For a given point P, $\Omega(n \log n)$ steps are required by any algebraic decision tree that can decide if there is an equitable two-fan for S with center at P.*

Proof: Let T be an algebraic decision tree that can decide for a set S with $\Theta(n)$ data points, and a point P, whether there is an equitable 2-fan for S centered at P. We will establish the lower bound on the height of T in the special case where a constant fraction of the points of S are fixed. It is clear that the lower bound for this special case is no more than the lower bound for the general case where no point of S is fixed. We will show that the subset X of *restricted* inputs where T returns a **NO** answer has at least $n!$ path connected components.

We take $P = (0, 0)$ and data points $Q_i = (r_i \cos(\theta_i), r_i \sin(\theta_i))$, $r_i > 0$. For each such point we only need its argument $\theta \in (0, 2\pi)$, since this alone determines whether the point is in some 2-fan centered at P, and we can take all $r_i = 1$. Inputs will consist of $N = 16n - 8$ points, described by the components of $\underline{z} = (z_0, \ldots, z_{N-1}) \in R^N$. Points z_j are blue if $j = 0$ or $7 \bmod 8$, red if $j = 1, 4, 5$ or $6 \bmod 8$, and green if $j = 2$ or $3 \bmod 8$. Thus each input describes a set S with $8n - 4$ red points, $4n - 2$ blue points, and $4n - 2$ green points.

The canonical input is the point $\underline{z}^* = (\theta_0, \ldots, \theta_{N-1}) \in R^N$, where

$$\theta_i = \frac{\pi}{2} + \left(\frac{i+1}{N+1}\right) 2\pi, \quad i = 0, \ldots, N. \tag{1}$$

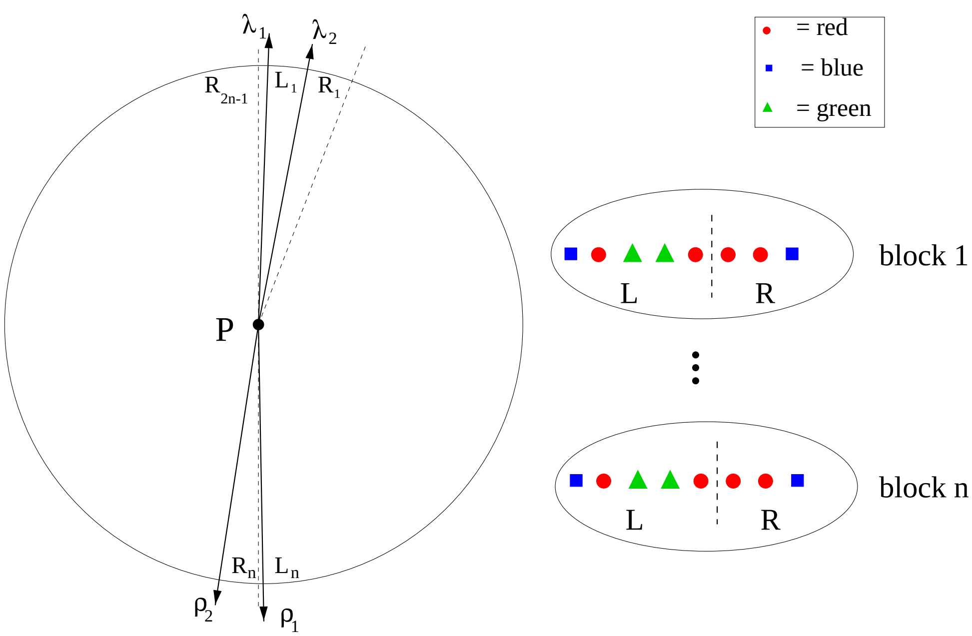

Figure 1. The canonical input $\underline{z}^*$.

These $16n - 8$ points are divided into $2n - 1$ blocks of 8 points each. The blocks have a LEFT part and a RIGHT part. Block i has left part $L_i = [\theta_{8i-8}, \theta_{8i-7}, \theta_{8i-6}, \theta_{8i-5}, \theta_{8i-4}]$ and right part $R_i = [\theta_{8i-3}, \theta_{8i-2}, \theta_{8i-1}]$ for which, replacing the entries by their colors, is $L_i = [b\, r\, g\, g\, r]$ and $R_i = [r\, r\, b]$.

Let λ_1 be a ray separating R_{2n-1} and L_1, and ρ_1, a ray separating L_n and R_n. This two-fan has $n - 1$ blocks and the left half of block n. Also it contains $4n - 2$ red points (half), $2n-1$ blue points (half) and $2n$ green points (one more than half). The next (clockwise) two-fan that equitably partitions the red and the blue points is λ_2 (a ray separating L_1 and R_1) and ρ_2 (a ray separating R_n and L_{n+1}) but now there are $2n-2$ green points (one less than half).

In fact ALL two-fans that split both red and blue points evenly are either deficient by one green or in excess by one green. Specifically, for each $j = 1, \ldots, n$, we have the two-fans with rays λ_{2j-1} and ρ_{2j-1} containing blocks $[L_j\, R_j\, \cdots\, R_{j+n-2}\, L_{j+n-2}]$ ($2n$ green points); also for each $j = 1, \ldots, n$, we have the two-fans with rays λ_{2j} and ρ_{2j} containing blocks $[R_j\, L_{j+1}\, \cdots\, L_{j+n-1}\, R_{j+n-1}]$ ($2n-2$ green points). Thus, with input $\underline{z}^*$, T must return a NO answer to the equitable two-fan query.

We consider only a restricted set I of inputs to T. A point $\underline{z} = (z_0, \ldots, z_{N-1}) \in I$ if $z_j = \theta_j$ for $j \neq 2 \bmod 16$, and otherwise $z_j \in (0, 2\pi]$. Only the first green point in the odd numbered blocks is free to vary; the rest are the relevant θ's. Each $\underline{z} \in I$ describes an input set of N points, $6n-3$ of them fixed, specific points, while the other $2n-1$ points are arbitrary on the unit circle. Write X as the set of restricted inputs in which T returns a NO answer.

For a permutation $\underline{\pi} = (\pi_1, \ldots, \pi_n)$ of $(1, \ldots, n)$, define $\underline{z}_\pi \in I$ by

$$z_{16j-14} = \theta_{16\pi_j-14};$$

$\underline{z}_\pi$ describes the same N points as $\underline{z}^*$ except that the first green points of the odd-blocks of $\underline{z}^*$ appear in permuted order in $\underline{z}_\pi$. Therefore $\underline{z}_\pi \in X$ for every permutation. We also claim that if $\underline{\pi}$ and $\underline{\rho}$ are distinct permutations, then $\underline{z}_\pi$ and $\underline{z}_\rho$ are in different path connected components of I. We move along a continuous path p in I from $\underline{z}_\pi$ to $\underline{z}_\rho$ (holding z_j fixed, $j \neq 2 \bmod 16$). As we do, some green point leaves its half-block, say L_k, and moves to an adjacent half-block, R_{k-1} or R_k. Let j be the block where this first occurs on p and let $\underline{z}(t) \in p$ denote the corresponding point in R^N. The restricted input described by this point is no longer in X because either $[R_{k-n-2}\, L_{k-n-1}\, \cdots\, L_{k-1}\, R_{k-1}]$ or $[R_k\, L_{k+1}\, \cdots\, L_{k+n-1}\, R_{k+n-1}]$ are now equitable partitions of all three colors. Therefore X has $n!$ path connected components, and the lemma now follows by Ben-Or's theorem [19]. ■

The claimed lower bound for finding an equitable 2-fan is somewhat misleading: it may be possible in $o(n \log n)$ time to determine a point Q, and two lines incident at Q for which one of the four 2-fans is equitable. Lemma 1 just says that given only Q, $n \log n$ steps are needed to know if there is an equitable 2-fan with apex Q, and to find one if YES. Thus any search algorithm (like Bereg's) that tests candidate apex points P must have complexity at least $n \log n$.

3 Other Equitable Partitionings in R^2

Given n points in general position in R^2, Willard [29] asked for a pair of non-parallel lines ℓ_1 and ℓ_2 that equitably partition the points; i.e., in each of the four open quadrants they define, there are at most $n/4$ points. An efficient algorithm for this was implied by results in Cole, Sharir, and Yap [15], and an optimal $O(n)$ algorithm follows immediately using Megiddo's separated, discrete ham-sandwich cut [25]. In fact we can even insist that the lines are orthogonal: this is proved in Courant and Robbins [16], or implied by a result of Bárány and Matoušek [8]) that uses Borsuk-Ulam along with equivariant topology. Here we give an easy, direct combinatorial proof of existence, along the lines of [16]. From this argument we are led to an efficient orthogonal partitioning algorithm. Specifically we prove

Theorem 1 *Given a set S of n points in general position in R^2, there exist orthogonal lines ℓ_1 and ℓ_2 that equipartition S, and they may by found in $\Theta(n \log n)$ RAM steps.*

Proof: (Existence argument) A halving line for S has at most $|S|/2$ points in its open halfspaces. For the existence, w.l.o.g. we may assume $n = 4j + 1$ is odd and start with ℓ_1 as a vertical halving line incident with the point $P \in S$ with median $x-$coordinate, and ℓ_2 as a horizontal halving line incident with the point $Q \in S$ of median $y-$coordinate. Also suppose the open upper left quadrant has the most points, say a, which we assume is $> j$ or this partition is already equitable. We will rotate ℓ_1 and ℓ_2 counter clockwise through $\pi/2$ radiands, always keeping them orthogonal, and keeping ℓ_1 a halving line: i.e., ℓ_1 rotates about P until it first meets a point - say $P_1 \in S$. Next we rotate about P_1 until ℓ_1 meets meets another point, P_2, etc. Except for the moments when ℓ_1 is incident with two points, P_i and P_{i+1}, it is always a halving line for S. During this process, as ℓ_2 passes points of S, we will move it (always maintaining its orthogonality to ℓ_1) so at most half the points of S are in either of *its* open halfspaces. At the end of the rotation the upper-left quadrant has become the original lower left, and now has $< j$ points (because ℓ_2 is halving). Since its cardinality changes by ± 1 at each "event" in the rotation, there is a position where it has exactly j points. ■

The complexity statement follows from the following two results.

Lemma 2 *Given a set S of $\Theta(n)$ points in general position in R^2 and a point $Q \in R^2$, $\Omega(n \log n)$ steps are required by any algebraic decision tree that can decide if there is an equitable partitioning of S by orthogonal lines incident with Q.*

Proof: The argument is a construction sharing several features with that of Lemma 1, so we will be terse. We take Q to be the origin. Let $N = 32k + 9$ and take N points on the unit circle with arguments given by

$$\theta_j = \frac{2\pi j}{N},\ j = 1, \ldots, N;$$

since N is odd, no two are antipodal. The points of S will be those θ_j where $j = 1, 2, 3, 4, 5 \bmod 8$, so $n = |S| = 20k + 5$. (It may help to think of $4k + 1$ groups of equally-spaced points, 8 points per group, plus one extra point. Each open quadrant has $k + 1/4$ groups. Points of S occupy the first 5 places in a group; the last 3 are empty.)

If two orthogonal lines through $Q = (0, 0)$ equipartition S there can be at most $5k + 1$ points in any open quadrant. Let ℓ_1 and ℓ_2 be orthogonal lines through Q. It is easy to see that

1. as they are rotated about Q (maintaining orthogonality), if *neither* is incident with a point of S, then exactly two quadrants each contain $5k+2$ points of S, and the other two contain a total of $10k+1$ points;
2. in addition, as either ℓ_1 (or ℓ_2) rotates across $\theta_i, \ldots, \theta_{i+7}$, there is a position where the four open quadrants contain $5k+2$, $5k+2$, $5k+1$, and $5k$ pts.

Thus, at the canonical input

$$\underline{z}^* = (\theta_1, \ldots, \theta_5, \theta_9, \ldots, \theta_{13}, \ldots, \theta_{32k+1} \ldots, \theta_{32k+5}) \in R^{20k+5},$$

the decision tree must answer NO. On the other hand take k even and consider the restricted set of inputs I where $z_j \in (0, 2\pi)$, and $z_i = \theta_i$ for $i \neq 11 \bmod(16)$. For each $\underline{\pi} = (\pi_1, \ldots, \pi_{2k})$, a permutation of $(1, \ldots, 2k)$, define $\underline{z}_\pi \in I$ by

$$z_{16j-5} = \theta_{16\pi_j - 5};$$

$\underline{z}_\pi$ and $\underline{z}_\rho$ are in different connected components because on a continuous path $p(t)$ in I from $\underline{z}_\pi$ to $\underline{z}_\rho$, one of the middle points in an even numbered group is first to enter an adjacent group, and at the input described by that $p(t)$, there is an allignment of ℓ_1 and ℓ_2 that is incident with a point of S, and where each open quadrant has at most $5k+1$ points of S, a YES input. ■

As with Lemma 1, it may be possible in $o(n \log n)$ time to determine orthogonal lines ℓ_1, ℓ_2 that equipartition the n given data points. The Lemma just says that given only Q, $n \log n$ steps are needed to know if there exist orthogonal lines incident at Q which do the job, so any search-based algorithm that tests candidate points must have complexity at least $n \log n$.

Lemma 3 *Given a set S with n points in general position in R^2, in $O(n \log n)$ RAM steps we can find orthogonal lines ℓ_1 and ℓ_2 that equitably partition the points.*

Proof: Dualizing the existence proof in Theorem 1, there is a point $P_1 = (-a, y_1)$, $a > 0$, that is dual to ℓ_1, and a point $P_2 = (1/a, y_2)$, dual to ℓ_2 ($\ell_2 \perp \ell_1$), so that (i) at most half of the lines in $\mathcal{L}$ (the set of lines dual to the n points in S) are above (below) P_1, (ii) at most half are above (below) P_2, and (iii) $\lfloor n/4 \rfloor$ are above *both* P_1 and P_2. We will search for P_1, starting with a point $Q = (-c, d)$ on the median level, and $c > 0$ chosen so that Q is to the left of the vertex of $A(\mathcal{L})$ with min x-coordinate. The cost to obtain Q is $O(n \log n)$ and we test it in $O(n)$ time by computing $Q' = (-1/c, d')$ on the median level, and counting $N(Q)$, the number of lines above *both* Q and Q', stopping with $P_1 = Q$ if $N(Q) = \lfloor n/4 \rfloor$.

We could now use slope selection to carry out a binary search on the vertices of $A(\mathcal{L})$ and after $\log \binom{n}{2}$ search steps, each evaluating $N(\cdot)$ at the relevant point, we determine P_1 at a cost of $O(n(\log n)^2)$. On the other hand in linear time we can find vertical lines $x = t_1$ and $x = t_2$ with the property that P_1 is in the strip they determine, but there are at most εn^2 vertices of $A(\mathcal{L})$ that are also within the strip, $\varepsilon > 0$ small. To do this we evaluate N at a point on $x = t_1$ and the median level and also at a point on $x = t_2$ and the median level, and accept (t_1, t_2) if N is bigger than $n/4$ at one point and smaller at the other. Now it is easy to prune a constant fraction of the lines of $\mathcal{L}$ which cannot determine P_1 and then recursing within the strip on $\mathcal{L}'$, the unpruned lines, we obtain P_1 in time $O(n \log n)$: the strip in the next recursive step is determined in time $O(n)$, because we evaluate $N(\cdot)$ with respect to the original lines, and there are $O(\log n)$ steps before only a constant number of lines remain, after which we can finish in $O(n)$ time by brute force search. ■

An interesting special case is where the n points of S are in convex position, so their radial order is the same from every point interior to $\operatorname{conv}(S)$. This fact simplifies the problem enough to allow us to find an orthogonal equipartitioning in linear time.

Lemma 4 *An equitable orthogonal partitioning for n points in convex position can be found in time $O(n)$.*

Proof: We may consider the points to be on the unit circle. The idea is a simple prune and search. We begin with any partition formed by orthogonal halving lines, ℓ_1 and ℓ_2. Let $a > n/4$ be the size of the largest quadrant, say the northwest as in the existence argument. Instead of rotating ℓ_1 and ℓ_2 counterclockwise we proceed using $O(\log n)$ binary search steps. The first one, for example, finds the point $Q \in S$ in the northwest quadrant that has the median argument, and constructs ℓ_1' as the halving line through it, and then ℓ_2' as the halving line orthogonal to ℓ_1', all performed in $O(n)$ time. If the "new" northwest quadrant has a quarter of the points, we are done. If it has $a' > n/4$ points, we continue counterclockwise; otherwise we have jumped too far with ℓ_1', ℓ_2' and so now we continue the search clockwise. In both these cases, $a/2$ points may be removed from further consideration.

After each such step, done in linear time, we find a new pair of orthogonal halving lines and can decide if the search continues in the counter clockwise direction, or not. At this point at least 1/8 of the points may be removed from further consideration and the search continues on the remaining points. The ability to "prune" the constant fraction at each step inplies that the entire cost is linear. The details are omitted. ■

Next, we consider the equitable partitioning induced by three lines incident at a point. Buck and Buck proved that for any continuous measure on a compact set, three such lines may be found for which each sector has a 1/6 fraction of the measure, and it is straightforward to extend this to an equipartitioning for points in the plane. Here we show

Theorem 2 *Given a set S of n points in general position in R^2, in $\Theta(n \log n)$ RAM steps we can find lines ℓ_1, ℓ_2, and ℓ_3, incident at a common point, and no sector contains more than $n/6$ points of S. If the points are in convex position the six-way partitioning may be found in linear time.*

Proof: Because the ideas are so similar to what was used for orthogonal equipartitioning, we omit many details. As before, the existence argument drives the formulation of an efficient algorithm, and suggests the framework for prune-and-search in the case of convex position.

Let ℓ_1 be any halving line. There is a unique line ℓ_2 that splits the points of S in *both* halfspaces in a "two-to-one" ratio; i.e., two opposite quadrants each contain $n/6$ points and the other two contain $n/3$ each. Notice that both lines are halving.

Let $Q = \ell_1 \cap \ell_2$ and consider the unique line ℓ_3 that is a (separated) ham-sandwich cut for the points of S in the two quadrants with $n/3$ points. We rotate this structure counterclockwise preserving (i) the fact that ℓ_1 is halving, (ii) the fact that ℓ_2 is two-to-one splitting, and (iii) the fact that ℓ_3 halves the points in each of the ℓ_1, ℓ_2 quadrants that contain $n/3$ points.

As in Theorem 1 we rotate the entire structure about a point $P \in S$ that is incident with ℓ_1. When ℓ_1 meets another point of S, that new point becomes the center of rotation. "Events" occur when any line passes a point of S. If ℓ_2 crosses points of S, we move it so it is still halving. If ℓ_3 crosses points of S in one of the ℓ_1, ℓ_2 quadrants that contains $n/3$ points we move it so it is still halving for that quadrant. Let $A = \ell_1 \cap \ell_3$ and $B = \ell_2 \cap \ell_3$. The punchline is that after rotation through π radiands, σ, the sign[1] of triangle ΔABQ, becomes $-\sigma$ and since the rotation is continuous and σ changes by ± 1, there is a moment when $\sigma = 0$, and the three lines pass through Q.

We will again use slope selection to guide a binary search on the vertices of the arrangement of lines dual to points in S. ℓ_1 begins as the vertical line through the point of S with median $x-$coordinate (or in the dual, the vertex on the median level with smallest $x-$coordinate), found in $O(n \log n)$. ℓ_2 and ℓ_3 may now be found in $O(n)$, along with the vertices of the triangle determined by these three lines, and σ, its sign. The next binary search step uses slope selection to find (in the dual) the vertex with median $x-$coordinate (say $x = t$) and then chooses the new ℓ_1 as the line on the median level with this x coordinate. Now the new ℓ_2 and ℓ_3 are determined in $O(n)$ time and the new value of σ: if it is zero, we are done; if it has the same sign as the previous configuration, we continue the search to the right, and otherwise to the left. There are $O(\log \binom{n}{2})$ steps, each at a cost of $O(n \log n)$ for a total cost of $O(n(\log n)^2)$.

We can reduce the overall cost to $O(n \log n)$ by doing the $O(\log n)$ slope selections approximately, getting more and more accurate as the search narrows in on the solution, as in the optimal slope selection algorithm [14]. For example the first binary search step only needs to return a vertex with $x-$coordinate of rank less than $3n^2/4$ but greater than $n^2/4$ and this can be done in linear time. In this fashion the total binary search cost is $O(n \log n)$, details omitted.

We also omit a construction, similar in conception to Lemma 2, showing that $\Omega(n \log n)$ steps are necessary for any algebraic decision tree that can decide if a set S with $\Theta(n)$ points in R^2 admits an equitable partitioning by three lines through a given point $Q \in R^2$. When the points of S are in convex position, the binary search steps can be done in linear time by ordinary selection in the primal. In addition points may be pruned after each search step, allowing an overall linear time algorithm for the

[1]Given distinct points $C, D, E \in R^2$, the sign of triangle ΔCDE is 1 if C is to the left of the directed line from D to E, -1 if C is to the right, and 0 if the points are collinear. The sign is also 0 if the points are not distinct.

convex case. ■

Finally, we consider the equitable partitioning induced by a convex quadrilateral and the two lines through its opposite vertices, the "cobweb" partitioning discovered by Schulman [27]. Each of the eight open regions of this configuration has at most $n/8$ points of S.

Theorem 3 *Given a set S of n points in general position in R^2, in $O(n \log n)$ RAM steps we can find a cobweb partitioning.*

Proof: (Brief outline) The upper bound is based on Schulman's existence proof, which already had an algorithmic flavor. From a starting construction with two halving lines and five sides of a "pseudo-quadrilateral", the algorithm either ends successfully, or rotates the structure to the next position. By the time the first diagonal has rotated into the original position of the second diagonal, a solution must have been discovered. As in the previous cases we can carry out the search via binary search, guided by an approximate slope selection which becomes more accurate as the solution is approached. ■

We believe that the convex case is linear and that the general algorithm is optimal.

4 Some Other Partitioning Problems

There are many fascinating questions of a similar flavor to the ones considered so far. Some concern the algorithmic aspects of partitionings whose existence is guaranteed by Borsuk-Ulam, as most of the previous results. Others pertain to the existence (or not) of a particular structure.

First, consider the following equipartitioning assertion (A_d) concerning a set S of n points in general position in R^d:

(A_d) **there exist d, $(d-1)$-dimensional hyperplanes in R^d, so that none of the 2^d "orthants" they determine have more than $n/2^d$ points of S**

A_1 just says there is always a median, a point splitting S into $n/2$ smaller items and $n/2$ bigger ones. A_2 say that a two-line equipartitioning exists for S (and in fact we know that these lines may be taken to be orthogonal). Yao et. al [30] proved A_3, guaranteeing the existence of three planes in R^3 that have at most $n/8$ points of S in any octant, and in fact one of the planes may be taken as an arbitrary halving plane. Finally for $d \geq 5$, Avis [5] showed the existence of sets so that no matter how d hyperplanes are chosen in R^d, there will always be at least $2^d - d^2$ orthants with NO points of S. The big open case is $d = 4$. Avis' result cannot guarantee any empty orthants and so the possibile truth of A_4 remains intriguingly open.

The $d = 3$ case is also interesting. The theorem in [30] allowed one of the three equipartitioning planes to be an arbitrary halving plane. It is not known if this extra degree of freedom may be used to prove the existence of a (1/8) equipartitioning with an extra property, say two of the planes having orthogonal normals, or one plane normal to the line common to the other two, etc.

Because this equipartitioning is so natural, Yao et. al. discussed the computational complexity. They discribed an $O(n^7)$ algorithm starting with any halving plane and then, using three points of S for each of the other two planes, iterated through the $\binom{n}{6}$ possibilities, testing each configuration in linear time, and knowing that one of them is guaranteed to work. This is already too slow for say, 50 points. The same idea can be refined, noticing that each of the other planes is a halving plane, and then using a recent bound of Sharir, Smorodinsky, and Tardos [28] on the number of three dimensional halving sets. This would imply an $O^*(n^6)$ algorithm, the * signifying the presence of log factors. In fact each of the other planes is a simultaneous halving plane for the points above the initial plane and for the points below it. We conjecture that there are $O(n^2)$ such planes and this now would give an $O^*(n^5)$ algorithm. More formally

Conjecture 1 *Let A and B be sets of n points in general position in R^3 with disjoint convex hulls. The number of halving sets of $A \cup B$ that simultaneously bisect both A and B is $O(n^2)$.*

Another familiar combinatorial consequence of Borsuk-Ulam is the necklace theorem [2], [3]: a strand of gems of d different types, $2n_i$ jewels of type i, may be equipartitioned with at most d cuts. The cuts create $d+1$ sub-strands which may then be partitioned into two groups in such a way that each group has n_i jewels of type i. Its clear that d cuts may be necessary, namely if the jewels of each type are consecutive; the theorem says d is always sufficient.

One nice (combinatorial) proof of this theorem represents the necklace as a sequence of $N = 2n_1 + \cdots + 2n_d$ points on the moment curve in R^d and then uses the ham-sandwich theorem. The computational complexity of this approach is $O^*(H_{d-1})$, by a result of Lo et. al. [23], H_j denoting the number of halving sets in R^j. For $d = 3$ types of jewels, the complexity is $O^*(n^{4/3})$ by virtue of Dey's bound on planar k-sets [17]. This should be compared to the $O(n^3)$ search of all possible 3 cuts. It would be interesting to know if there is a more efficient algorithm.

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