Instructions: **1.** Do all your work in the blue examination booklets. **2.** Present answers IN THE GIVEN ORDER, though you could solve questions in any order. You may use books and notes, but all work must be your own. **3.** Show *ALL* your work. I want to see HOW you get the answer much more than the answer itself, so you will get *little* or *no* credit for unexplained answers. **4.** There is no need to reduce expressions to simplest terms if all the numbers are there. **5.** The value of each question appears in parentheses. Use this as a guide in allocating your time. An asterisk (*) denotes a more challenging one, which you may wish to leave to the end. You have two and a half hours for the test which should be enough for the 131 unstarred points (there are also 19 * pts.).

1. This question uses the function

$$f(t) = 1/t, \; 1 \leq t \leq 4,$$

and collocation points $x_0 = 1$, $x_1 = 2$, and $x_2 = 4$.

(a) (12 pts) Write down *Lagrange's form* of $I_2(t)$, the quadratic interpolation of f based on x_0, x_1, x_2, above. Now use *this formula* to evaluate $I_2(3)$. For each arithmetic operation $+, -, *$, and $/$, count the number of those operations you performed and explain your counting.

(b) (10 pts) Write down the error formula describing the error of I_2 at t, and explain your answer. Then use this formula to bound the error at $t = 3$. Again, explain.

(c) (10 pts) Write down the system of equations you would need to solve to obtain $I_2(t)$ *in standard form*.

(d) (13 pts) Now add the collocation point $x_3 = 3$ and find c_3, the coefficient of the cubic term in Newton's form of $I_3(t)$, the degree ≤ 3 polynomial that interpolates f at x_0, x_1, x_2, x_3. How many multiplication and division steps did you use to obtain c_3? Explain how you obtained this number.

(e) (5 pts) Find $T_1(t)$ the degree ≤ 1 Taylor polynomial for f, expanded about $u = 2$.

(f) (10 pts) Use Taylor's theorem to express $f(t) - T_1(t)$, the error of T_1 at a point t. Explain. Now use this formula to bound the error at $t = 3$.

(g) (8 pts) Write down the collocation points that determine $C_2(t)$, the quadratic Tchebycheff interpolation of f on $[1, 4]$.

(h) (8 pts; 4 (*) pts) Find $M_1(t)$ linear minimax (or "best") approximation to f on this interval. (*) What is $d(f, M_1) = \max_{1 \leq t \leq 4} |f(t) - M_1(t)|$? Explain.

(i) (10 pts) Using $x_0 = 1$, $x_1 = 2$, $x_2 = 4$, and $x_3 = 4$ as evaluation points, find P_1, the discrete least-squares quadratic polynomial approximation to f, expanded in *the monomial basis.* Explain your work.

(j) *(5 pts) Decide if M_1 and C_1 are the same in this problem? Explain.

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2. The parts of this question are not related.

   (a) (10 pts) Let $f(x) = (1+x)^3$, $0 \leq x \leq 3$. Compute the symmetric difference approximation to the derivative of f at $x = 1$ using a stepsize of $h = .1$ Write down an expression for its error, carefully explaining the terminology.

   (b) (5 pts) Decide whether $\phi_0(x) = 1$, $\phi_1(x) = 2x$ are orthogonal basis functions on $[-2, 2]$ and explain your answer.

   (c) (10 pts) In a few CLEAR sentences answer: (i) What is the Hilbert matrix? (ii) Give an example of a real problem where it occurs; (iii) What computational problems may arise because of the Hilbert matrix, and how may they be overcome?

3. This question tests your geometric understanding of polynomial approximation methods. Very little computation should be needed. Consider the function

$$f(x) = \frac{1}{1+9x^2}, \quad -1 \leq x \leq 1$$

   and the following four approximations (also on $[-1, 1]$): $g(x) = 1$, $h(x) = 11/20$, $p(x) = 2/11$, $q(x) = 1/10 + 18(x+1)/100$.

   (a) (5 pts) Graph $f(x)$ and each of $g(x)$, $p(x)$, $h(x)$ and $q(x)$ (either all on the same plot, or each function along with f).

   (b) (5 pts) Which approximation is $T_1(x)$ the linear Taylor polynomial expanded about $u = 0$? Explain.

   (c) (5 pts) Which approximation is $C_1(x)$ the linear Tchebycheff interpolation? Explain.

   (d) (5 pts) Which approximation is $M_1(x)$ the linear minimax approximation? Explain.

   (e) *(10 pts) Is g or q the better approximation to f? Answer in terms of the distance $d(f, g) = \max\left(|f(x) - g(x)|, x \in [-1, 1]\right)$. Explain.