CS510 TEST 2 December 20xx

Instructions:

- **Do all your work in the blue examination booklets.**
- **WRITE ANSWERS IN THE GIVEN ORDER, though you can work on them in any order.**
- **NO CALCULATORS and NO CELLPHONES.**
- **You may consult Handout 4 on Differentiation and Integration along with ONE page of prepared notes (both sides, normal size writing). Otherwise, all work should be your own.**
- **Show ALL your work. You will get Little or No credit for an answer that is not explained.**
- **No need to reduce answers to simplest terms.**
- **The value of each question is indicated, 125 points in all. Use this as a guide in allocating your time - 2.25 hours for the exam.**

1. You have a computer that does $k = 4$ decimal-digit arithmetic. The problem is to compute X, the sum of n given inputs, $a_1, a_2, \ldots, a_n$. The algorithm is to begin with $S_1 = a_1$ and then, for each $j = 2, 3, .., n$, compute $S_j = S_{j-1} + a_j$ and finally return S_n as the computed value of X.

   (a) (8 pts) Suppose $a_1 = 1001$ and for $i = 2, \ldots, n$, $a_i = .4 + 1/i$. What answer does our computer give under the *chopping* convention? What answer under the rounding convention? Finally describe the relative errors of BOTH these computations, assuming n is very large, say $> 2^{10,000}$.

   (b) (7 pnts) Suggest an algorithm (no double-precision) that could prevent large roundoff errors - as above - for this problem, and explain (i) WHY it works and (ii) if it always gives the best-possible answer for the given inputs.

2. You want to learn the value of π. The method is to compute INT, the integral of

$$f(x) = 2/(1 + x^2)$$

   over the interval $[-1, 1]$ ($INT = \pi$ because $\int_{-t}^{t} f(x)dx = 4 \arctan(t)$, and $\tan(\pi/4) = 1$).

   (a) (5 pts) Approximate INT using the composite midpoint rule with $n = 2$ subdivisions. Be explicit about how you get the numerical value of the approximation.

   (b) (7 pts) Write down an expression for the error of the above approximation and then use it to get an upper bound for the magnitude of the error (the smaller the upper bound the better).

(c) (8 pts) Assume now that we do NOT KNOW explicitly what f is, but just have access to its value at x via a computer program. "If we use the forward difference approximation to $f'(1/2)$ on our computer (it uses $k-$digit arith) the approximations will converge to $-32/25$ as the stepsize converges to zero". Decide if the statement in quotes is TRUE or FALSE and back up your assertion with some relevant facts.

3. The parts of this question are unrelated.

   (a) (10 pts) The cubic Taylor polynomial approximation for $f(x) = 12e^x$, expanded about $u = 0$ is
   $$T_3(x) = 12 + 12x + 6x^2 + 2x^3,$$
   and $T_3(2)$ is its approximation of $f(2) = 12e^2$. Compute this approximation (i.e., $T_3(2)$) **(by hand)** **as efficiently as you can**, making clear exactly how you are doing it. How many $''+''$ or $''-''$ operations were used? How many $''*''$ ops? How many $''/''$ ops?

   (b) (15 pts) Now let $f(x) = x^2$ and take *data points* $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$. Find $P_1(x)$, the degree $k = 1$ discrete least squares approximation to f, expanded in the *monomial basis*. Explain What you are doing, and why. Then decide if there could be another polynomial $Q_1(x) \neq P_1(x)$ that is as good an approximatio to f as is P_1 - based on the given data - and explain your answer.

   (c) (10 pts) Find orthogonal basis functions ϕ_0, ϕ_1, ϕ_2 for polynomials of degree at most two on the interval $[-2, 5]$.

   (d) (15 pts) The function $h(x) = (4x - x^2)/3$ has a fixed point at $w = 1$.

      i. First verify that FPI on h starting with $P_0 = 2$ gives $P_1 = 4/3$ and $P_2 = 32/27$.

      ii. Next compute P_2^*, the accelleration of P_2.

      iii. Now do the next FPI step, first assuming you are following Aitkins algorithm, and then repeating that step, but now assuming you are following Steffanson's algorithm. Explain your work.

      iv. (*) Is it justified to apply accelleration to this FPI? Explain your answer.

   (e) (10 pts) In a few CLEAR sentences answer: (i) What is the Hilbert matrix H_n? (ii) Give an example of a real problem where it occurs; (iii) What computational problems may arise because of the Hilbert matrix? What may be done to circumvent them?

4. This question tests your geometric understanding of polynomial approximation methods. Very little computation should be needed. Consider the function
   $$f(x) = \frac{1}{1 + 9x^2}, \quad -1 \leq x \leq 1$$
   and the following four approximations (also on $[-1, 1]$): $g(x) = 1$, $h(x) = 11/20$, $p(x) = 2/11$, $q(x) = 1/10 + 18(x + 1)/100$.

   (a) (5 pts) Graph $f(x)$ and each of $g(x)$, $p(x)$, $h(x)$ and $q(x)$ (either all on the same plot, or each function along with f).

   (b) (5 pts) Which approximation is $T_1(x)$ the linear Taylor polynomial expanded about $u = 0$? Explain.

   (c) (5 pts) Which approximation is $C_1(x)$ the linear Tchebycheff interpolation? Explain.

   (d) (5 pts) Which approximation is $M_1(x)$ the linear minimax approximation? Explain.

   (e) (10 pts) Is g or q the better approximation to f? Answer in terms of the distance $d(f, g) = \max(|f(x) - g(x)|, x \in [-1, 1])$. Explain.