Midterm, October 26, 2015 - Some Solutions

• **Question 2a:** $S = \{(h_1, \ldots, h_{15})$, where $h_i \in \{1, \ldots, 15\}$ is the hat person $i$ gets, and $h_i \neq h_j$ if $i \neq j\}$. This is the set of all permutations of the hats, one to a person. $|S| = 15!$.

• **Question 2b:** $|A| = 14 \cdot 14!$ and $|B| = 13!$: there are 14 choices for which hat person 1 gets (he cannot get his own) and for each, 14! ways to distribute the other hats; once the hats of one and two are exchanged, there are 13! ways to distribute the remaining hats. Note that $B \subset A$.

• **Question 2c:** Because $B \subset A$, $P_B(A) = 1$ but $P(A) < 1$ so they are NOT independent.

• **Question 2d:** There is ONE way to give each woman her own hat, and 10! ways to distribute the remaining hats, so $|C| = 10!$ and $P(C) = 10!/15! = 1/(15)^5$.

• **Question 2e:** Again, there is ONE way to give each woman her own hat and (let’s say) $d_{10}$ ways to derange the other 10 hats (so none of the 10 owners get their own). Recall that $q_{10} = 1 - 1/! + 1/2! - 1/3! + \cdots + 1/10!$ is the probability that you get derangement in the 10 hat experiment and that $q_{10} = d_{10}/10!$, so $d_{10} = 10! \cdot q_{10}$ and $P(D) = q_{10}/(15)^5$.

• **Question 3a:** $S$ is the collection of ALL subsets of the 100 senators of size 15 and $|S| = \binom{100}{15}$, $P(A) = \frac{\binom{50}{15} \cdot 2^{15}}{|S|}$; we choose 15 of the 50 states and from each, one of the two senators.

• **Question 3b:** You can pick the two states in $\binom{50}{2}$ ways. We then need to pick eleven of the remaining forty eight states and from each, one of the two senators. Thus $|B| = \binom{50}{2} \cdot \binom{48}{11}$. $A$ and $B$ cannot be independent because $A \cap B$ is empty (if $A$ occurs, $B$ cannot). This means that $P_A(B) = 0 \neq P(B) = |B|/|S|$.

• **Question 4a:** $S = \{(b_1, \ldots, b_{15})$, where $b_i \in \{1, \ldots, 16\}$ is the BOX that ball $i$ goes into$, |S| = 16^{15}$.

• **Question 4b:** $|A| = 16$, the number of choices for WHICH box all balls will go into; $|B| = (16)_{15}$.

• **Question 4c:** $|C| = \binom{15}{3} \cdot 13! \binom{15}{2} + \binom{15}{2} \cdot 15! \binom{13}{2} 11!$: we can choose the 3 boxes that will be empty in $\binom{16}{3}$ different ways. When we place the 15 balls in the remaining 13 boxes so that NONE of them are empty, either $D$ (some box has three balls and the other 12 have one each) or $E$ (a PAIR of the 13 boxes get TWO balls each and the other 11 boxes get ONE ball each) must occur. $|D| = 13! \binom{15}{3} 12!$ (13 choices for WHICH BOX gets three balls, $\binom{15}{3}$ ways to choose WHICH BALLS all go in that box, and 12! ways to place the remaining balls, one in each empty box among the 13 that will not be empty at the end). $|E| = \binom{15}{2} \binom{15}{2} \binom{13}{2} 11!$. 
