1. Consider the linear system

\[ A' = (A|\bar{b}) = \begin{pmatrix} 1 & 3 & | & 4 \\ -2 & 4 & | & 2 \end{pmatrix}. \]

(a) (9 pts) Find \( A^{-1} \) using Gauss-Jordan elimination with NO row interchanges. Then use it to find the solution to \( Ax = \bar{b} \). Explain all your steps.

- We reduce \( A \) to the identity \( I \) by row operations. The same sequence applied to \( I \) produces \( A^{-1} \). Here are the steps:

\[
A = \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 \\ 0 & 10 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I
\]

so doing the same ops on the identity gives

\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} \frac{4}{10} & \frac{3}{10} \\ 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} \frac{4}{10} & \frac{3}{10} \end{pmatrix} \rightarrow A^{-1}.
\]

You might verify that \( A^{-1}A = I \).

(b) (10 pts) Find the LUP factorization of \( A \) using Gaussian elimination with partial pivoting, then back-solving for the solution. Again, explain what you are doing.

- The partial pivoting strategy chooses row \( m \) to eliminate \( x_j \) from ALL equations below equation \( j \) if \( m \geq j \) and if \( |c_{m,j}| = \max |c_{i,j}|, i \geq j > 0 \). In our problem partial pivoting would choose equation 2 as the pivot for \( x_1 \) so initially \( L = I \), the identity, \( U \) is \( A \) with its two rows exchanged, and \( p \) is the vector \((2, 1)\) as a column.

We now do the row operation \( \text{row}_2 < -\text{row}_2 + (1/2)\text{row}_1 \) and record the pivot value \(-1/2\) in \( \text{row}_2 \) of \( L \) and we end with the factorization

\[
L = \begin{pmatrix} 1 & 0 \\ -1/2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -2 & 4 \\ 0 & 5 \end{pmatrix}, \quad p = \begin{pmatrix} 2 \\ 1 \end{pmatrix}
\]

(c) (9 pts) In this part, \( A' = (A|\bar{b}) \) is now a system of \( n \) linear equations in \( n \) unknowns where the coefficients satisfy (i) \( a_{ij} = 0 \) if \( i + j \leq n \) and (ii) \( a_{ij} \neq 0 \) if \( i + j = n + 1 \).

Argue that the system has a unique solution. How much work (the number of \(*\) and \(/\) steps used) is needed to find it? Explain how you got your answer.

- The coefficient matrix is

\[
A = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 & a_{1,n} \\ 0 & \cdots & \cdots & \cdots & \cdots & a_{2,n} \\ 0 & 0 & 0 & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & \cdots & a_{n-1,n} & a_{n,n} \end{pmatrix}
\]

Once you understand the structure of \( A \) you easily see that there is a unique solution: \( x_n = b_1/a_{1,n} \) and the denominator is not zero. After that, use forward substitute (the value of \( x_n \) in eq. 2) to see that \( x_{n-1} = (b_2 - a_{2,n} \ast x_n)/a_{2,n-1}, \) and again, the denominator is not zero so all is “legal” etc., etc. Each such forward substitution step will determine a new component of the unique solution \( x \).
2. We seek the roots of \( f(x) = e^{x-1} - x \).

(a) (4 pts) Show that \( w = 1 \) is the only root of \( f \).

- Notice that \( f(1) = e^0 - 1 = 0 \) so \( w = 1 \) is a root of \( f \). Also \( f'(x) = e^{x-1} - 1 \) and so \( f'(1) = 0 \), showing that the root of \( f \) at \( x = 1 \) is a tangency root. Finally \( f''(x) = e^{x-1} > 0 \) for all \( x \) showing that the slope of \( f \) is increasing with \( x \). These facts show that as \( x \) increases, \( f \) decreases to zero at \( x = 1 \) and then increases with \( x \), and at a rate that also increases with \( x \). These facts show that \( w = 1 \) is the unique root of \( f \), and that \( f > 0 \) except at \( x = w \).

(b) (6 pts) Will Newton’s method converge if you start at a value \( P_0 \) that is close enough to 1? Explain. If “YES”, at what rate?

- Newton’s method will monotonically converge to 1 whether you start at \( P_0 < 1 \), or \( P_0 > 1 \). The convergence rate will be linear in both cases - tangency root.

(c) (5 pts) Can you use bisection to find the root? Explain.

- You cannot use bisection to find the root because \( f \) is always non-negative, a property that violates the starting conditions for bisection.

(d) *(5 pts)* Describe what will happen if you use FPI on \( g(x) = e^{x-1} \).

- If you start at \( P_0 < 1 \) FPI will converge monotonically to \( w = 1 \), but if you start at \( P_0 > 1 \) it will diverge to infinity, and at an increasing rate (successive steps \( |P_{n+1} - P_n| \) increase with \( n \)).

(e) (10 pts) Get three regula-falsi approximations to the root of \( f(x) = x^3 - 4 \) starting with the interval \((a, b) = (1, 2)\) and then accelerate the last one.

- You should show you know what a r.f. step is \([if on (a, b) \text{ we have } f(a) * f(b) < 0 \text{ then } P \leftarrow (a * f(b) - b * f(a))/(f(b) - f(a)) \text{ and we stop if } f(P) = 0 \text{ or continue with } (a, P) \text{ or } (P, b) \text{ whichever has } f \text{ opposite signs at the endpoints. I got } P_0 = 1.42857, P_1 = 1.55046, \text{ and } P_2 = 1.57916 \text{ and applying the acceleration formula } (P_2') = (P_2 - P_1)^2/(P_2 - 2 * P_1 + P_0) \text{ i got } P_2' = 1.58800\).

3. The following graph shows a function \( h(x) \) and the line \( y = x \). As the graph suggests, \( h''(x) > 0, a_2 < x < a_4 \). Also \( h(a_1) = h(a_3) = h(a_5) = h'(a_2) = h'(a_6) = 0 > h'(a_5) > -1 \) and \( h(a_i) = a_i \), when \( i = 2, 4, 5, 7 \).

NO PIC HERE

(a) (15 pts) We will do fixed point iteration on \( h \) \([P_{n+1} = h(P_n)]\) starting at \( P_0 = a_4 + \epsilon \), where \( \epsilon > 0 \) is much smaller than \( a_5 - a_4 \). Will it converge? If YES, to what value, and at what rate? Explain. If NO, what happens? Explain. Repeat for \( P_0 = a_4 - \epsilon \), where \( \epsilon > 0 \) is much smaller than \( a_4 - a_3 \). Can FPI reach \( x = a_4 \)? Explain.

- YES it converges to \( a_5 \). The rate is linear because \( h'(a_5) < 0 \).
  - At \( a_4 - \epsilon \) FPI will converge to \( a_2 \) and at a quadratic rate (at least) because \( g' \) is zero there.
    - FPI CAN reach \( x = a_4 \) as follows: if at \( P_n \) we have \( h(P_n) = a_4 \) then the next step is \( P_{n+1} = a_4 \).

(b) (5 pts) Repeat (a), now starting at \( P_0 = a_6 \).

- \( P_1 \) will be 0 and from there the iterations converge at least quadratically fast to \( a_2 \).
(c) (5 pts) Now we will use Newton's method on $h$ starting at $p_0 = a_3 + \epsilon$, where $\epsilon > 0$ is much smaller than $a_4 - a_3$. Will it converge? If YES, to what value, and at what rate? Explain. If NO, what happens? Explain.

- Newton will converge to $a_3$ and at least a quadratic rate - it’s a non-tangency root.