CS206 **HW2c SOLUTIONS** **Dec. 10, 2017**

There are solutions for MOST of the required problems.

- Hand-in solutions to questions marked by a (*) by Dec. 7, 2017.
- Problems marked by (**) are more challenging, and required for students in the Honors section.
- Problems marked by (+) are NOT required for hand-in and wont be graded, but are probably interesting.
- NOTICE hard question (e) is now NOT REQUIRED!

1. The bridge experiment (model 1) is to choose 13 cards from the deck at random (use equally likely probability).
   (a) What is the probability you get NO aces? What is the probability you get no aces, but you have all the Kings? Whats the conditional probability you got no aces, given that you got all the Kings?
   (b) (*) What is the probability of $A = \{$you get ALL the aces$\}$ but NO Jacks? What is the probability of $B = \{$you get all the aces and the remainimg cards are all spades? What is the conditional probability of B, given A?
      - The sample space is $S = \{$ALL subsets of 13 cards from the deck$\}$ and its size is $|S| = \binom{52}{13}$. If you must get all 4 aces there are 9 other cards you need from the remaining 48, and the prob is $\binom{48}{9}/|S|$. But if you cannot get Jacks the remaining 9 cards are chosen from the 44 cards (that are neither Aces nor Jacks), and the probability is $P(A) = \binom{44}{9}/|S|$.

        If you get all 4 aces and the remaining cards must be spades the prob. is $P(B) = \binom{12}{9}/|S|$; you need 9 cards that must be spades but you already have the ACE, so there are only 12 left in the deck.
   (c) Find the conditional probability of no aces given that you got NO spades.
   (d) (*) What is the probability you get NO spades? What is the probability you get no card higher than 9?
      - Let $A = \{$no spades$\}$ and $B = \{$all cards $\leq 9\}$. Its easy to see that $P(A) = \binom{39}{13}/\binom{52}{13}$ and $P(B) = \binom{32}{13}/\binom{52}{13}$; counting ACE as high there are eight card values not larger than 9 and thus 32 cards to choose from for event B.
   (e) (**) What is the probability *neither* of the two events in (d) [above], occurs?
      - "NEITHER A NOR B from d) is a complex event: it is $A^c \cap B^c = (A \cup B)^c$ so the $P(\text{neither}) = 1 - P(A \cup B)$; I will just compute $P(A \cup B)$. It is $P(A \cap B^c) + P(B)$, and since $P(B)$ was already determined, I just count $|A \cap B^c|$.

        Of the 39 non-spades, 15 are greater than nine and 24 are less than 9, so an outcome in $A \cap B^c$ must have one or more of its 13 non-spades greater than 9 and the rest must be less than 9. THESE facts together show that $P(A \cap B^c) =$

        $$[\binom{15}{1}\binom{24}{12} + \binom{15}{2}\binom{24}{11} + \cdots + \binom{15}{13}]/\binom{52}{13},$$

from which we easily get the required answer.

2. (*) The bridge experiment (model 2) is a <u>*partition experiment*</u> where (i) 13 cards are chosen at random for the first player (East), (ii) 13 of the remaining cards are chosen at random for the second player (South), (iii) 13 of the remaining cards are chosen at random for the third player (West), and (iv) the remaining cards are given to the last player (North). Repeat question 1 <u>all parts [(a)-(d)]</u> in this model, and, where you are East; i.e., player 1. Is it different if you are West, that is, player 3?

   - I will just compute the probability that when NORTH is dealer and deals the cards to East then South, then West then North (either one card at a time or a 13 card group to each player going around the table, or any other way to make the partition), that WEST has NO Aces but ALL four KINGS; i.e., the event A. The sample space is the set of partitions of the deck into four hands, each of size 13 for East, South, West, North. The cards in each hand are unordered, but the hands are ordered E,S,W,N. We know $|S| = \binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}$. I will count how many of the elements in S are in the event A. If West is to be dealt 4 Aces and 9 other cards - none of them Kings - from the 44 cards that are neither Ace nor King - he must get one of the $\binom{44}{9}$ subsets of those cards that make up A. If this is to happen, East must be dealt 13 of the remaing 39 cards (cards not to be dealt to WEST), South, 13 of the remaining 26 cards and North the final 13. The probability is $\binom{44}{9}/\binom{52}{13}$, just as it was in part a)!!! [NOTICE we are just counting the partitions that are consistent with WEST getting a certain hand.]

3. (*) There are 10 pairs of shoes in a closet. Five shoes are picked at random. What is the probability there is no "pseudo-pair" in the sample (i.e., one left and one right shoe)? What is the probability that there is no pair?

   - $|S| = 20_5$ taking the experiment as an ordered sample without replacement from the shoes. There is no pseudo pair (event A) only if all five picks are left shoes, or if all are right. The first pick has 20 possibilities. Once made, the other four picks have to agree with the first pick as to L/R, so $|A| = 20 \cdot 9 \cdot 8 \cdot 7 \cdot 6$.

     If there is to be no pair (event B) there are 20 possible first picks, 18 second picks (you must not pick the mate of the first shoe, so we discard it before the next pick), etc., and thus $|B| = (20 \cdot 18 \cdot 16 \cdot 14 \cdot 12$.

4. (+)(**) An ordinary deck of 52 cards is shuffled. What is the probability that no aces are adjacent? What is the probability that no spades are adjacent?

   - You can order the 48 non-aces in 48! ways. For our event, each ace must go in a different "space", 47 of them *between* successive non-aces, one *before* the first non-ace, and one after the last non-ace. That leaves 49 different "spaces" into which a single ace could be placed, only one ace to a space, and in this way, no aces could be adjacent. This event can be achieved in $48!\binom{49}{4}4!$ different ways [permute the non-aces; select four different gaps for the aces; order the aces in those gaps, left-to right], and there are 52! orderings of the deck, so the probability is $(49_4)/(52_4)$.

     The same idea applies to the probability that all spades are separated by at least one card. The probability is $40_{13}/52_{13}$.

5. (+) As above a deck of cards is shuffled. What is the probability that no aces are adjacent or separated by only one non-ace?

6. (+) 4 cards are randomly dealt to each of 13 players.

   (a) Describe the sample space and write down its size.

      - The sample space S is the set of partitions of a 52-card deck into 13 ordered hands of four cards each, and its size is $|S| = \binom{52}{4}\binom{48}{4}\cdots\binom{4}{4} = 52!/[4!]^{13}$.

   (b) What is the probability of $A = \{$each player has one card from each suit$\}$?

      - $|A| = (13!)^4/|S|$. The first player has 13^4 choices of which card he gets from each suit, the second has 12^4, etc.

   (c) What is the probability of $B = \{$each player has all four cards of the same value$\}$?

      - $|B| = 13!$

   (d) (*) Compute $P_B(A)$ and $P_A(B)$. Are A and B independent?

      - $P_B(A) = 1$ - if B occurs each players four cards are the same value (say KING) so each has all four suits in his hand. The events are NOT independent.

        $P_A(B) = |A \cap B|/|A| = 13!/(13!)^4 = 1/(13!)^3$.

   (e) What is the probabiility that players one, two, and three have been dealt only aces, kings, or queens. As usual, explain.

      - The 12 Aces Kings, Queens are given to the first three players in $\binom{12}{4} * \binom{8}{4}$ ways and the remaining cards dealt in $40!/(4!)^{10}$ ways.

   (f) (**) What is the probability that *one* player has one card from each suit but that nobody else has cards from *more* than one suit?

      - I may do this later.

7. A fair die is tossed twice. Let $X =$ the sum of the faces, $Y =$ the maximum of the two faces, and $Z = |\text{face } 1 - \text{face } 2|$.

   (a) Write down the value of X,Y, Z, and $W = XZ$ for each outcome $w \in S$.

   (b) Find $Range(X)$, $Range(Y)$, $Range(Z)$, and $Range(W)$.

   (c) Describe the partitions $\mathcal{A}_X$ and $\mathcal{A}_Z$ induced by these random variables.

   (d) Find f_X, f_Y, f_Z, and f_W, the frequency functions.

   (e) Are the events $A = \{w \in S : X(w) = 7\}$ and $B = \{w \in S : Z(w) = 1\}$ independent?

8. (*) A fair coin is tossed four times. Consider the following random variables on S, the sample space: $X =$ the number of Heads; $Y =$ the length of the longest block of successive Tails (0 if NO Tails); $Z =$ the number of the toss on which the last Tail occurred (0 if NO Tails); $W = \max(X, Y)$; $V = \min(X, Z)$. Use equally likely probability on S.

   (a) List the elements of S. For each $w \in S$, write the value of each random variable.

   (b) Find the *Range* of each random variable.

   (c) Find the frequency function of each random variable and plot it.

      - These questions are ALL straightforward and tedious. I may get to do some.

   (d) Describe the partitions induced by X and W.

   (e) Are the events $A = \{w : X(w) = 2\}$ and $B = \{w : Y(w) = 2\}$ independent? Explain.