1. In experiment $E$ we toss a fair coin four times in succession. Consider the following random variables on the sample space $S$: $X = \text{the number of Tails}$; $Y = \text{the length of the longest block of successive Heads (0 if NO Heads)}$; $Z = \text{the number of the toss on which the last Tail occurred (0 if NO Tails)}$; $U = \max(X, Y)$; $V = \min(X, Z)$. Use equally likely probability on $S$.

(a) List the elements of $S$. For each $w \in S$, write the value of each random variable.

(b) Find the Range of each random variable.

(c) Find and plot the frequency function of each random variable.

(d) Describe the partitions induced by $X$ and $V$.

(e) Are the events $A = \{w \in S : X(w) = 2\}$ and $B = \{w \in S : Y(w) = 2\}$ independent? Explain.

2. (*) A fair die is tossed twice. Let $X = \text{the sum of the faces}$, $Y = \text{the maximum of the two faces}$, and $Z = |\text{face 1} - \text{face 2}|$.

(a) Write down the value of $X$, $Y$, and $W = XZ$ for each outcome $w \in S$.

(b) Find $\text{Range}(X)$, $\text{Range}(Y)$, and $\text{Range}(W)$.

(c) Describe the partitions $A_X$ and $A_Z$ induced by these random variables.

(d) Find $f_X$, $f_Y$, $f_Z$, and $f_W$, the frequency functions.

(e) Are the events $A = \{w \in S : X(w) = 7\}$ and $B = \{w \in S : Z(w) = 1\}$ independent?

3. The experiment $E$ is to choose at random a rooted binary tree with 3 nodes. Let $X$ be the number of leaves and $Z$, the height of the tree (longest path [\# of edges] from the root to a leaf). Use equally likely probability on the sample space of $E$.

Find the frequency functions $f_X$ and $f_Z$ and decide if these random variables are independent.

4. (*) As above, but now we choose a FOUR node rooted binary tree.

5. An experiment $E$ has sample space $S = \{1, 2, 3\}$ with probabilities $1/6, 2/6, 3/6$, respectively. $E$ is performed twice (trial 1 and trial 2) and $S^{(2)} = S \times S = \{w = (w_1, w_2) : w_i \in S\}$ is the outcome of the $i^{th}$ trial} is the composite sample space. We will use the following probability measure $P$ on $S^{(2)}$:

$$
\begin{array}{c|cccccccc}
 w \in S^{(2)} & (1, 1) & (1, 2) & (1, 3) & (2, 1) & (2, 2) & (2, 3) & (3, 1) & (3, 2) & (3, 3) \\
\end{array}
$$

Consider random variables $X = \text{the result of the first trial and $Z = \text{two times the result of the second trial, minus one; therefore $X(w_1, w_2) = w_1$ and $Z(w_1, w_2) = 2w_2 - 1.$}$}

(a) Check that $P$ is “product probability”.

(b) Show that these probabilities on $S^{(2)}$ “respect” the probability on $S$; this means that if $A_i$ denotes the event that you get $i$ on the first trial and $B_j$, the event you get $j$ on the second trials, then $P(A_i) = i/6$ and $P(B_j) = j/6$. 

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(c) Find Range(Z), $A_Z$, the partition induced by Z, and $f_Z$, the frequency function of Z.
(d) Are $X$ and $Z$ independent?