1. In experiment $\mathcal{E}$ we toss a fair coin four times in succession. Consider the following random variables on the sample space $S$: $X =$ the number of Tails; $Y =$ the length of the longest block of successive Heads (0 if NO Heads); $Z =$ the number of the toss on which the last Tail occurred (0 if NO Tails); $U = \max(X, Y)$; $V = \min(X, Z)$. Use equally likely probability on $S$.

(a) List the elements of $S$. For each $w \in S$, write the value of each random variable.

(b) Find the Range of each random variable.

(c) Find and plot the frequency function of each random variable.

(d) Describe the partitions induced by $X$ and $V$.

(e) Are the events $A = \{w \in S : X(w) = 2\}$ and $B = \{w \in S : Y(w) = 2\}$ independent? Explain.

2. (*) A fair die is tossed twice. Let $X =$ the sum of the faces, $Y =$ the maximum of the two faces, and $Z = |\text{face 1} - \text{face 2}|$.

(a) Write down the value of $X, Y, W = XZ$ for each outcome $w \in S$.

(b) Find $\text{Range}(X)$, $\text{Range}(Y)$, and $\text{Range}(W)$.

(c) Describe the partitions $A_X$ and $A_Z$ induced by these random variables.

(d) Find $f_X$, $f_Y$, $f_Z$, and $f_W$, the frequency functions.

(e) Are the events $A = \{w \in S : X(w) = 7\}$ and $B = \{w \in S : Z(w) = 1\}$ independent?

3. The experiment $\mathcal{E}$ is to choose at random a rooted binary tree with 3 nodes. Let $X$ be the number of leaves and $Z$, the height of the tree (longest path [\# of edges] from the root to a leaf). Use equally likely probability on the sample space of $\mathcal{E}$.

Find the frequency functions $f_X$ and $f_Z$ and decide if these random variables are independent.

4. (*) As above, but now we choose a FOUR node rooted binary tree.

5. An experiment $\mathcal{E}$ has sample space $S = \{1, 2, 3\}$ with probabilities $1/6, 2/6, 3/6$, respectively. $\mathcal{E}$ is performed twice (trial 1 and trial 2) and $S^{(2)} = S \times S = \{(w_1, w_2) : w_i \in S\}$ is the outcome of the $i$th trial} is the composite sample space. We will use the following probability measure $P$ on $S^{(2)}$:

<table>
<thead>
<tr>
<th>$w \in S^{(2)}$</th>
<th>(1, 1)</th>
<th>(1, 2)</th>
<th>(1, 3)</th>
<th>(2, 1)</th>
<th>(2, 2)</th>
<th>(2, 3)</th>
<th>(3, 1)</th>
<th>(3, 2)</th>
<th>(3, 3)</th>
</tr>
</thead>
</table>

Consider random variables $X =$ the result of the first trial and $Z =$ two times the result of the second trial, minus one; therefore $X(w_1, w_2) = w_1$ and $Z(w_1, w_2) = 2w_2 - 1$.

(a) Check that $P$ is “product probability”.

(b) Show that these probabilities on $S^{(2)}$ “respect” the probability on $S$; this means that if $A_i$ denotes the event that you get $i$ on the first trial and $B_j$, the event you get $j$ on the second trials, then $P(A_i) = i/6$ and $P(B_j) = j/6$. 

(c) Find Range(Z), \(A_Z\), the partition induced by Z, and \(f_Z\), the frequency function of Z.

(d) Are \(X\) and \(Z\) independent?

6. (*) \(E\) is the hat check experiment with \(n = 4\) hats. Let \(X\) count the number who do not get their own hat and let \(Z\) be the indicator of the event \(A = \{\) neither person 1 nor person 3 gets their own hat\(\}\) (i.e., \(Z(w) = 1\) if \(w \in A\) and \(Z(w) = 0\) if \(w \notin A\).) We use equally likely probability on the sample space \(S\). Repeat (c) and (d) of the previous question.

7. Players \(A\), \(B\), \(C\) and \(D\) toss a fair coin in order (i.e., \(A\), then \(B\), then \(C\), then \(D\), then \(A\) again, etc). The first player to get a Head wins. What are their respective chances to win, using product probability on the sequences of tosses?

8. (***) Repeat the above with players \(A\), \(B\), and \(C\) who now toss a pair of fair dice and win if they are first to get a sum 7 or 11 on their toss.

9. Experiment \(E\) is to toss two fair dice. This experiment is repeated \(n\) times. Find the (product) probability \(P^{(n)}(A)\) of the event that you get at least one trial with a sum of 7. What is the smallest \(n\) for which \(P^{(n)}(A) > .5\).

10. Each day the stock price moves up one point or down one point with probabilities \(1/4\) and \(3/4\) respectively (these are hard times). What is the probability that after 4 days the stock will be at its original price (assume the daily changes are independent)?

11. You are playing ping-pong with a friend and your chance to win any point is \(P\).

   (a) (*) Find the probability that you score 4 points before your friend has a score of 4. Evaluate this expression for \(P = 1/2\) and \(P = 2/3\). Any comments? (this is also a model for the world series: i.e.; “best of seven”).

   (b) Repeat the above except you must score 3 points before your opponent scores 3 (this is a “best-of-five” series).

   (c) Do the above “best-of-three” problem but NOW given the fact that you win the first game.

12. The experiment \(E\) is to take a required computer science course. The outcomes are the grades \(\{A, B^+, B, C^+, C, D, F\}\). Let \(E^{(5)}\) be the experiment “do \(E\) 5 times”, once for each of CS111, CS112, CS113, CS205, and CS206. Assume the grades are equally likely.

   (a) Let \(X\) count the number of distinct grades you receive in the five course sequence. Compute the frequency function \(f_X\).

   (b) (*) As above, but just compute \(f_X(1)\) (all five grades are the same), but now assuming that \(A, B^+, C, D\), are each given with probability 1/10, and that \(B, C^+, F\) are each given with probability 2/10.

13. A fair die is tossed twice and we use equally likely probability on the sample space. Let \(X\) be the score of the first toss and \(Y\), the score of the second. Show (i.e., “prove”) that \(X\) and \(Y\) are independent.