Please solve both problems below.

**Problem 1.** Suppose \( f : \{0, 1\}^n \to \{0, 1\} \) is a function and let \( \mu \) be a distribution over \( \{0, 1\}^n \). We define the **average query complexity** of \( f \) over the distribution \( \mu \) as:

\[
D_\mu(f) := \min_{\text{algorithm } A} \text{average number of queries of } A \text{ to } x,
\]

where each query of \( A \) to \( x \in \{0, 1\}^n \) simply asks for the value of \( x_i \) for a given \( i \); here, the average in the query complexity is taken over the choice of \( x \sim \mu \).

Define \( f^m : (\{0, 1\}^n)^m \to \{0, 1\} \) where

\[
f^m(x^1, \ldots, x^m) = (f(x^1), f(x^2), \ldots, f(x^m)).
\]

The goal of this question is to prove a lower bound for \( D_\mu^m(f^m) \) based on \( D_\mu(f) \), i.e., a direct sum result for average query complexity of \( f \). (\( x^1, \ldots, x^m \sim \mu^m \) is sampled by picking each \( x^i \) independently from \( \mu \).)

Formally, we like to prove that

\[
D_\mu^m(f^m) \geq m \cdot D_\mu(f).
\]

(i) Let \( A \) be any algorithm for \( f^m \) with probability of success \( 2/3 \) and average query complexity \( q \) over \( \mu \). Define the following algorithm \( B \) for \( f \) over \( x \sim \mu \):

(a) Sample \( i \in [m] \) uniformly at random and set \( x^i = x \).
(b) Sample \( x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^m \) independently from \( \mu \).
(c) *Simulate* running \( A \) over \( (x^1, \ldots, x^m) \) and output the same answer that \( A \) outputs for \( x^i \) in \( f^m \).

Show how to do the simulation and implement \( B \) in a way that it achieves a probability of success \( 2/3 \) for \( f \) over \( \mu \), while having average query complexity \( q/m \).

(ii) Use part (i) to prove the direct sum result.

**Problem 2.** Define **Echo** as the following communication problem: Alice gets a single bit \( x \in \{0, 1\} \) and Bob gets no input; the goal for Bob is to output \( x \), i.e., “echo” \( x \). Consider the distribution \( \mu \) which is uniform over \( \{0, 1\} \). Clearly, **Echo** requires 1 bit of communication for \( x \sim \mu \) to success with probability more than half.

(i) Prove that (external) information complexity of **Echo** over the distribution \( \mu \) (with probability of success \( 2/3 \)) is also \( \Omega(1) \).

*Hint:* Use Fano’s inequality.

(ii) Use part (i) plus the direct sum of external information complexity for one-way protocol to prove that information complexity of the Index problem over uniform distribution on \( \{0, 1\}^n \) and \( i \in [n] \) is \( \Omega(n) \).

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\(^{1}\)It is easier to work with the average ‘cost’ of the algorithm in this problem compared to the typical worse-case cost. However, one can easily transition between the two by a small cost in query cost and probability of success of the algorithm.