I. (i) Using Gaussian elimination with complete pivoting, obtain a $PAQ = LU$ decomposition of

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

and use it to solve

$$Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$ 

(ii) Use the above decomposition to determine $A^{-1}$.

II. Obtain an $LDL^T$ factorization of

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 4 & 11 & 4 \\ 2 & 4 & 6 \end{bmatrix}.$$ 

III. As a prelude to this question, note that matrix multiplication can be viewed in terms of submatrices. For example, if $A_{m \times n}$ and $B_{n \times p}$ are partitioned as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where, for $i = 1, 2$ and $j = 1, 2$, $A_{ij}$ is $m_i \times n_j$ and $B_{ij}$ is $n_i \times p_j$, then

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}.$$ 

(i) Assume $A$ is a nonsingular $n \times n$ matrix whose triangular factors $LU = A$ are known, and that we wish to compute the analogous factors $L, U$ of an $(n+1) \times (n+1)$ matrix of the form

$$B = \begin{pmatrix} A \alpha \\ \beta^T \gamma \end{pmatrix}$$

where $\alpha$ and $\beta$ are $n$-vectors, and $\gamma$ is a scalar. Describe how $L$ and $U$ can be used for this purpose. How many additional operations are needed? [Hint: Write $L, U$ in the form

$$L = \begin{pmatrix} L \\ \lambda^T \end{pmatrix}, \quad U = \begin{pmatrix} U \mu \\ 0 \nu \end{pmatrix}.$$]
(ii) Let $A_k$ denote the $k$-th principal submatrix of an $n \times n$ matrix $A$, i.e.,

$$A_k = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}.$$ 

One way to compute the LU factorization of $A$ would be to start with that of $A_1$ and compute, in succession, those of $A_2, \ldots, A_n$ via the procedure in part (i). How many operations are needed to compute $L$ and $U$ in this way?

IV. Let $A, B$ be $n \times n$ matrices, $A$ nonsingular, and $x$ an $n$-vector. How many operations are needed to compute the quantities below? [Avoid computing $A^{-1}$ in (ii) and (iii) - it’s inefficient.]

(i) $(AB)^n x$
(ii) $A^{-1} B x$
(iii) $A^{-1} B$

V. Consider the linear system $Ax = b$ in question I.

(i) What are $\|A\|_{\infty}$, $\|A^{-1}\|_{\infty}$, and $\kappa_{\infty}(A)$?
(ii) Suppose $\hat{x}$ is an approximate solution to $Ax = b$ such that $\|A\hat{x} - b\|_{\infty} \leq .001$. Bound the absolute and relative error in $\hat{x}$ ($\|\hat{x} - x\|_{\infty}$ and $\|\hat{x} - x\|_{\infty}/\|x\|_{\infty}$).

VI. For the matrix $A$ in question I, find righthand side vectors $b$ and $b + \delta b$ such that equality is achieved in the condition number bound as applied to $Ax = b$ and $A(x + \delta x) = b + \delta b$: 

$$\frac{\|\delta x\|_{\infty}}{\|x\|_{\infty}} = \kappa_{\infty}(A) \frac{\|\delta b\|_{\infty}}{\|b\|_{\infty}}.$$ 

VII. Suppose $A$ is nonsingular and consider the pair of linear systems:

$$Ax = b, \quad (A + \delta A)(x + \delta x) = b.$$ 

Show that 

$$\frac{\|\delta x\|}{\|x + \delta x\|} \leq \kappa(A) \frac{\|\delta A\|}{\|A\|}.$$ 

Thus $\kappa(A)$ measures the sensitivity of the solution to $Ax = b$ with respect to changes in $A$ as well as in $b$.

VIII. Suppose we’ve computed the solution $x$ of $Ax = b$, saving the LU factors of $A$, and that we need to solve a new system $A'x' = b$ where $A' = A - uv^T$. Here $u, v$ are vectors; thus $uv^T$ is a ‘rank one’ matrix - it has only one independent row (or column).

(i) Show that $x'$ can be written in the form

$$x' = x + \alpha A^{-1} u$$
where $\alpha$, a scalar, is given by
\[ \alpha = \frac{v^T x}{1 - v^T A^{-1} u}. \]

(ii) Suppose $A'$ differs from $A$ only in its $k$-th column. How would you choose $u, v$? How many operations are needed to compute $x'$?

IX. Consider the problem of approximating a function $f(x)$ over $x \in [a, b]$ in the least squares sense by a linear combination of the form
\[ y(x) = \sum_{j=1}^{n} c_j \phi_j(x). \]
Here $\{\phi_j(x)\}_{j=1}^{n}$ are user-defined “basis functions,” chosen according to the type of approximation sought. For example, to generate a polynomial of degree $p$, we might use “monomial” basis functions
\[ \phi_j(x) = x^{j-1}, \quad j = 1, \ldots, n, \quad n = p + 1, \]
(an obvious - but very bad - way to represent a polynomial). We seek the set of coefficients $c_j$ which minimizes
\[ E = \int_{a}^{b} (y(x) - f(x))^2 \, dx \]
over all functions $y(x)$ of the given form. Substituting for $y(x)$ in the above, and setting the partial derivatives $\frac{\partial E}{\partial c_i} = 0$, $i = 1, \ldots, n$, we obtain the “normal equations” for our least squares approximation problem:
\[ \sum_{j=1}^{n} \left( \int_{a}^{b} \phi_i(x) \phi_j(x) \, dx \right) c_j = \int_{a}^{b} f(x) \phi_i(x) \, dx, \quad i = 1, \ldots, n, \]
which we write in matrix form as $Ac = b$.

For the case where the approximation interval is $[0, 1]$ and the monomial basis is used, we get as our coefficient matrix $A$ the “Hilbert” matrix:
\[ a_{i,j} = \frac{1}{i + j - 1}, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n, \]
a classic ill-conditioned matrix. Compute least squares approximation of $f(x) = 1 - x$ by polynomials of degree 3, 6, 9, 12 using the monomial basis. For each approximation, print the corresponding computed coefficients $\hat{c}$, the relative error $\|\hat{c} - c\|_\infty$, and the $\infty$-norm condition number of $A$. Use Matlab, and the very convenient commands hilb, norm, cond, diary; also ’c = A\b’ to solve the system, ’format long’ to print your results to full machine precision. Type ’help hilb’, ’help norm’, etc. to find out how to use these commands.

Interpret your results. To what extent is $\kappa(A)$ a good indicator of the number of digits lost in your computed solutions?