Additional problems + solutions

I. In Romberg integration, we kept doubling the number of subintervals and getting new trapezoidal rule approximations to a given integral $I$. Suppose instead of doubling, we repeatedly triple the number of subintervals...

(i) How can $T(h)$ be obtained from $T(3h)$ without performing any unnecessary integrand evaluations?

(ii) Derive an error estimate for $T(h)$ in terms of $T(h)$ and $T(3h)$.

II. In class, we found that the most accurate 2-subinterval trapezoidal rule approximation to $\int_0^1 \sqrt{x} \, dx$ uses subintervals $[0, 1/4]$ and $[1/4, 1]$ and that the error in each of these subintervals is the same. For $\int_0^1 x^2 \, dx$...

(i) What is the most accurate 2-subinterval trapezoidal rule approximation?

(ii) For the approximation in (i), is the error the same in each subinterval?

III. Derive a numerical differentiation formula of the form

$$f'(a) \approx w_0 f(a) + w_1 f(a + h) + w_2 f(a + 2h)$$

based on quadratic interpolation of $f(x)$ at $x = a, a + h, a + 2h$. Obtain a convenient expression for the error term for this formula.

IV. For the (disconnected) 6-node web with links \{12, 23, 24, 31, 41, 43, 56, 65\}...

(i) What is the corresponding directed graph $G$?

(ii) Take $\alpha = .85$ and use Matlab to compute the corresponding PageRank vector $r$:

(a) using the basic iteration for PageRank.

(b) via Gaussian elimination.

(iii) What happens to the definition of PageRank in the limit $\alpha \to 1$? What happens to the computational algorithms you used in (ii)?

Solutions

I. Romberg integration, with tripling of subintervals.

(i) 

$$T(h) = \frac{h}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_n-1) + f(x_n))$$

$$= \frac{3h}{3} (f(x_0) + 2f(x_3) + 2f(x_6) + \cdots + 2f(x_n-3) + f(x_n))$$

$$+ h (f(x_1) + f(x_2) + f(x_4) + f(x_5) + \cdots + f(x_n-2) + f(x_n-1))$$

$$= \frac{1}{3} T(3h) + h \sum \text{‘new’ } x_i$$
(ii) 
\[ \frac{I - T(h)}{I - T(3h)} = \frac{-\frac{(b-a)h^2}{12} f''(\xi)}{-\frac{(b-a)(3h)^2}{12} f''(\xi')} \approx \frac{1}{3^2} \]

Solving for \( I \)... 
\[ I \approx T(h) + \frac{1}{8} (T(h) - T(3h)) \]

II. 
\[
\int_0^1 x^2 \, dx = \int_0^{x_1} x^2 \, dx + \int_{x_1}^1 x^2 \, dx
\]
\[
= \left[ \frac{x_1}{2} (0 + x_1^2) - \frac{x_1^3}{12} f''(\xi_1) \right] + \left[ \frac{1-x_1}{2} (x_1^2 + 1) - \frac{(1-x_1)^3}{12} f''(\xi_2) \right]
\]
\[
|\text{error}| = \frac{1}{6} (x_1^3 + (1-x_1)^3)
\]
\[
\frac{d}{dx_1} |\text{error}| = \frac{1}{2} (x_1^2 - (1-x_1)^2) = x_1 - \frac{1}{2} = 0
\]

minimum at \( x_1 = \frac{1}{2} \) ← optimal value

For \( x_1 = \frac{1}{2} \), the approximation to \( \int_0^1 x^2 \, dx \) is \( \frac{3}{8} \), and the error is the same, \( \frac{1}{48} \), in each subinterval.

III. Take \( a = 0 \):
\[ f'(0) \approx w_0 f(0) + w_1 f(h) + w_2 f(2h) \]

and impose exactness for \( f(x) = 1, x, x^2 \ldots \)
\[
f(x) = 1 : w_0 + w_1 + w_2 = 0
\]
\[
f(x) = x : hw_1 + 2hw_2 = 1
\]
\[
f(x) = x^2 : h^2w_1 + 4h^2w_2 = 0
\]
solution: \( w_0 = -\frac{3}{2h}, \ w_1 = \frac{2}{h}, \ w_2 = \frac{1}{2h} \)

error term:
\[
(a - (a + h))(a - (a + 2h)) \frac{f^{(3)}(\xi)}{3!} = \frac{h^2}{3} f^{(3)}(\xi), \ \xi \in (a, a + 2h)
\]

final result:
\[ f'(a) = \frac{1}{2h} [-3f(0) + 4f(h) - f(2h)] + \frac{h^2}{3} f^{(3)}(\xi), \ \xi \in (a, a + 2h) \]
IV. $A = \begin{bmatrix}
0 & 0 & 1 & \frac{1}{2} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}$, $\mathbf{\bar{A}} = A$, $\mathbf{\bar{A}} = \alpha A + \frac{1-\alpha}{n} e e^T$

For any probability vector $p^{(0)}$, the iteration $p^{(k+1)} = \mathbf{\bar{A}}p^{(k)}$ converges to $r = (.2037, .1981, .1556, .1092, .1667, .1667)^T$.

As $\alpha \to 1$, limit of $p^{(k+1)} = \mathbf{\bar{A}}p^{(k)}$ either does not exist (e.g., if $p^{(0)} = (0, 0, 0, 0, 1)^T$) or will depend on $p^{(0)}$. Also, the coefficient matrix in (4) of pageRank notes becomes singular, and the system has no solution.