I. Consider the nonlinear equation \( f(x) = x^3 + x^2 - 1 = 0 \).

(i) Show that this equation has one and only one root in the interval \([0, 1]\).

(ii) Starting with the interval \([0, 1]\), compute \(x_0, x_1\) via the bisection method. Bound
the error in \(x_1\). How large must \(k\) be to obtain an iterate \(x_k\) with absolute error less than
\(10^{-6}\)?

(iii) Starting with the interval \([0, 1]\), compute \(x_0, x_1\) via regula falsi.

II. (i) Below are several different \(x = g(x)\) reformulations of the nonlinear equation in I.
For which of them will the corresponding fixed point iteration \(x_{k+1} = g(x_k)\) be locally
convergent to the solution \(\bar{x}\) in \([0, 1]\)? (The condition to check is whether \(|g'(\bar{x})| < 1\).)

(a) \(x = g(x) = \frac{1}{x^2} - 1\)
(b) \(x = g(x) = \frac{1}{\sqrt{x+1}}\)
(c) \(x = g(x) = x - \frac{1}{10}(x^3 + x^2 - 1)\)
(d) \(x = g(x) = x - \frac{x^3 + x^2 - 1}{3x^3 + 2x}\) (Newton’s method)

(ii) Use one of the above \(g's\) to compute \(\bar{x}\) to within absolute error \(10^{-4}\). [Suggestion:
Write a simple Matlab program to do this.]

III. For the nonlinear equation in I, show via the contraction mapping theorem that \(x_{k+1} = g(x_k)\ with g(x) as in II(b) will converge to the solution \(\bar{x} \in [0, 1]\). Will the rate of
convergence be faster than that of the bisection method? Explain.

IV. The cube root of 10 is the solution of \(f(x) = x^3 - 10 = 0\). Compute the cube root of
10 to within absolute error \(10^{-6}\) as follows using Matlab:

(i) Newton’s method with \(x_0 = 3\).

(ii) The secant method with \(x_0 = 2, x_1 = 3\).

V. Suppose \(x_{k+1} = g(x_k)\ is converging linearly to \(\bar{x}\), which we would like to compute to
within absolute error \(\epsilon\). We ask: Is it safe to stop iterating when \(|x_{k+1} - x_k| \leq \epsilon\, and then
use \(x_{k+1}\ as the approximation for \(\bar{x}\)? To answer this question, assume the conditions of
the contraction mapping theorem hold over an interval \(I\ containing \(\bar{x}\). Then

\[
|x_{k+1} - \bar{x}| \leq L|x_k - \bar{x}|, \quad L < 1 \tag{*}
\]

Using the following inequality

\[
|x_k - \bar{x}| \leq |(x_{k+1} - \bar{x}) + (x_k - x_{k+1})| \leq |x_{k+1} - \bar{x}| + |x_{k+1} - x_k|
\]

in (*), show that

\[
|x_{k+1} - \bar{x}| \leq \frac{L}{1 - L}|x_{k+1} - x_k|.
\]
error $\epsilon$. Will it be safe to take as the approximation the first iterate $x_{k+1}$ for which $|x_{k+1} - x_k| \leq \epsilon$? (Your answer will depend on $L$.)

VI. Let $p_n(x)$ denote the $n^{th}$ degree Taylor polynomial approximation for $f(x) = \exp(x)$ about $x = 0$. Write a Matlab program, 'Taylor.m', say, which evaluates $\exp(x)$ and $p_1(x)$, $p_2(x)$, $p_3(x)$ at $x = -1.0, -0.9, ..., 8.9, 1.0$. Your program should then:

(i) Graph all four sets of data on the same plot. [To do this, use Matlab commands: plot, xlabel, ylabel, and legend (to label the plots). You can find out how to use these commands via the Matlab 'help' facility - try typing 'help plot', for example.]

(ii) Compute $\max |p_n(x) - \exp(x)|$ over the given points for $n = 1, 2, 3$, and print these quantities. [An easy way to output a quantity is to omit the semicolon in the statement where it acquires its value.]