Decision Trees for Entity Identification: Approximation Algorithms and Hardness Results

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ABSTRACT

We consider the problem of constructing decision trees for entity identification from a given relational table. The input is a table containing information about a set of entities over a fixed set of attributes and a probability distribution over the set of entities that specifies the likelihood of the occurrence of each entity. The goal is to construct a decision tree that identifies each entity unambiguously by testing the attribute values such that the average number of tests is minimized. This classical problem finds such diverse applications as efficient fault detection, species identification in biology, and efficient diagnosis in the field of medicine. Prior work mainly deals with the special case where the input table is binary and the probability distribution over the set of entities is uniform. We study the general problem involving arbitrary input tables and arbitrary probability distribution over the set of entities. We consider a natural greedy algorithm and prove an approximation guarantee of $O(r_K \cdot \log N)$, where N is the number of entities, K is the maximum number of distinct values of an attribute, and r_K is a suitably defined Ramsey number. In addition, our analysis indicates a possible way of resolving a Ramsey theoretic conjecture by Erdös. We also show that it is NP-hard to approximate the general version of the problem within a factor of $\Omega(\log N)$.

General Terms

Decision tree, Approximation algorithm, Ramsey theory

Keywords

Entity identification, Decision tree, NP-hard

1. INTRODUCTION

Decision trees for the purposes of identification and diagnosis have been studied for a long time now [15]. Consider a typical medical diagnosis application. A hospital maintains a table containing information about diseases. Each row in the table is a disease and each column is a medical test and the corresponding entry specifies the outcome of the test for a person suffering from the given disease. Some of the medical tests are costly (e.g. MRI scans) and some require few days for the result to be known (e.g. blood cultures). When the hospital receives a new patient whose disease has not been identified, it would like to determine the shortest sequence of tests which can unambiguously determine the disease of the patient. Such a capability would enable it to achieve objectives like saving the expenditure of the patients, quickly determining the disease to start the treatment early etc. Motivated by such applications, we consider the problem of constructing *decision trees for entity identification* from the given data.

Decision Trees for Entity Identification - Problem Statement. The input is a table $\mathcal{D}$ having N rows and m columns. Each row is called an *entity* and the columns are the attributes of these entities. Additionally, we are also given a probability distribution $\mathcal{P}$ over the set of entities. For each entity e, $\mathcal{P}$ specifies $p(e)$, the likelihood of the occurrence of e. A solution is a decision tree in which each internal node is labeled by an attribute and its branches are labeled by the values that the attribute can take. The entities are the leaves of the tree. The main requirement is that the tree should identify each entity correctly. For an entity e, let $d(e)$ denote its distance from the root. The *weighted distance* of an entity e is given by $p(e)d(e)$. The sum of the weighted distances of all the entities is said to be the *weighted external path length* of the tree. The goal is to construct a decision tree with the minimum weighted external path length. We call this the $\mathcal{DT}$ problem. We call the special case of the $\mathcal{DT}$ problem in which the probability distribution over the set of entities is uniform as the $\mathcal{UDT}$ problem. For a given table, the maximum number of distinct values that any attribute may take is called its *branching factor*. We call the special case of $\mathcal{DT}$ in which every attribute takes on at most K distinct values, for a constant K, as the K-$\mathcal{DT}$ problem. Notice that any instance of the K-$\mathcal{DT}$ problem has a branching factor of at most K. The special case of the K-$\mathcal{DT}$ problem in which the probability distribution $\mathcal{P}$ is uniform is called as the K-$\mathcal{UDT}$ problem. 2-$\mathcal{DT}$ is the special case in which all the attributes are binary (i.e., $K = 2$). Figure 1 shows an example table, having a branching factor 5, and two decision trees for it. When the probability distribution is uniform, the weighted external path length of the first decision tree is $14/6$ and that of the second (and the optimal) decision tree is $8/6$.

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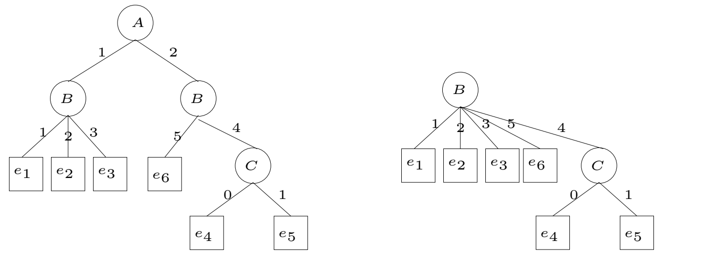
	A	B	C
e_1	1	1	1
e_2	1	2	0
e_3	1	3	1
e_4	2	4	0
e_5	2	4	1
e_6	2	5	0

Input Table with uniform probability distribution



A Decision Tree of cost 14/6 Optimal Decision Tree of Cost 8/6

Figure 1: Example decision trees

Previous results. Hyafil and Rivest [14] showed that the 2-$\mathcal{UDT}$ problem is NP-hard. Garey [6, 7] presented a dynamic programming based algorithm for the 2-$\mathcal{UDT}$ problem that finds the optimal solution, but the algorithm runs in exponential time in the worst case. Recently, Heeringa and Adler [13] proved the first non-trivial approximation ratio of $(1 + \ln N)$ for 2-$\mathcal{UDT}$ (see also [1, 12]). They also showed that it is NP-hard to approximate the 2-$\mathcal{UDT}$ problem within a ratio of $(1 + \epsilon)$, for some $\epsilon > 0$. Our analysis for the approximation ratios builds on that of Heeringa and Adler.

Our results. Much of the previous literature deals with the special case of 2-$\mathcal{UDT}$. We study the $\mathcal{DT}$ problem where the attributes can take multiple values with arbitrary probability distribution $\mathcal{P}$ over the set of entities. This occurs commonly, for example, in medical diagnosis applications (e.g. blood-group can take multiple values) and species identification (e.g. the leaf shape can be of many types). We present new results on both $\mathcal{DT}$ and $\mathcal{UDT}$ problems.

We present an $O(\log N)$-approximation algorithm for the 2-$\mathcal{DT}$ problem. We also show that it is NP-hard to approximate the 2-$\mathcal{DT}$ problem within a ratio of $\Omega(\log N)$. Thus, our results for the 2-$\mathcal{DT}$ problem are asymptotically optimal. For the general $\mathcal{DT}$ problem, we get an approximation ratio of $O(r_K \log N)$ where K is the branching factor of the table $\mathcal{D}$ and r_K is a suitably defined Ramsey number which is at most $(2 + 0.64 \log K)$.

For the $\mathcal{UDT}$ problem in which the probability distribution over the set of entities is uniform, we get an approximation ratio of $r_K(1+\ln N)$. We also show that it is NP-hard to approximate the $\mathcal{UDT}$ and the 2-$\mathcal{UDT}$ problems within ratios of $(4 - \epsilon)$ and $(2 - \epsilon)$ respectively, for any $\epsilon > 0$.

The following table summarizes our results where the only previously known result is shown in citation.

Problem	Apprx. Ratio	Hardness of Apprx.
2-$\mathcal{UDT}$	$1 + \ln N$ [13]	$(2 - \epsilon)$ for any $\epsilon > 0$
$\mathcal{UDT}$	$r_K(1 + \ln N)$	$(4 - \epsilon)$ for any $\epsilon > 0$
2-$\mathcal{DT}$	$O(\log N)$	$\Omega(\log N)$
$\mathcal{DT}$	$O(r_K \log N)$	$\Omega(\log N)$

Ramsey Numbers and Connections to Erdös' Conjecture. As mentioned earlier, our analysis of the greedy algorithm has interesting connections with Ramsey theory and an unresolved conjecture by Erdös. Ramsey theory, treated at length in the book by Graham et. al [10], deals with coloring the edges of complete graphs (or hypergraphs) with a specified number of colors satisfying certain constraints. For our purposes, we need the following specific type of Ramsey numbers.

Throughout the paper, we use n to denote the number of vertices in a graph and k to denote the number of colors. For $n > 0$, let G_n denote the complete graph on n vertices. A k-coloring of G_n is a coloring of the *edges* of G_n using k colors. For $k > 0$, R_k is defined to be the smallest number n such that any k-coloring of G_n contains a monochromatic triangle [1]. The inverses of the Ramsey numbers are more convenient for our purposes. For $n > 0$, we define r_n to be the smallest number k such that we can color the edges of G_n using only k colors without inducing any monochromatic triangle.

The exact values of the Ramsey numbers for $k > 3$ are not known. However, it is easy to show that for any k, $2^k \leq R_k \leq 1 + k!e$ (see [26]). A better lower bound is that for any k, $R_k \geq \frac{3^k+1}{2}$ [17, 25]. Even though better bounds have been proven [17, 4, 3] asymptotically, we do not need them for our discussions.

The lower bound on R_k translates into the following bound on the inverse Ramsey numbers r_n: for any n, $r_n \leq 2 + 0.64 \log n$. This shows that the greedy heuristic is a $(2+0.64 \log K)(1+\ln N)$-approximation algorithm for the K-$\mathcal{UDT}$ problem thus improving the ratio obtained by the black-box algorithm by a constant fraction. We note that any further improvement in the upper bound on r_n would automatically improve the approximation guarantee for the greedy procedure. Unfortunately, we cannot hope to get too much improvement, because of the following known lower bound: $r_n = \Omega(\log n/ \log\log n)$ [26]. Even proving a matching upper bound looks bleak in the light of a conjecture by Erdös (see [17]). The conjecture is that for some constant α, $R_n \leq \alpha^n$, for all n. Equivalently, the conjecture says that $r_n = \Omega(\log n)$. If the conjecture is true, via our approach, we can only hope for an improvement in terms of constant factors.

On the positive side, our results lead to an interesting approach to resolving Erdös' conjecture. Consider the K-$\mathcal{DT}$ problem. The idea is to construct a family of instances for which the cost of the greedy tree is more than that of the optimal tree by a factor of $\Omega(\log K \log N)$. More precisely, exhibit a constant c and a family of instances of the K-$\mathcal{DT}$ problem (one for each K) such that $|\mathcal{T}| \geq c \log K \log N |\mathcal{T}^*|$, where $|\mathcal{T}|$ and $|\mathcal{T}^*|$ refer to the cost of the greedy and optimal trees, respectively. Such a

[1]A monochromatic triangle is a triplet of vertices such that all the three edges between them have the same color. In Ramsey theory, R_k is denoted $R(3, 3, \ldots, 3)$, where "3" is repeated k times. For example, it is known that $R_1 = 3$, $R_2 = 6$, $R_3 = 17$ [22]

construction would imply the conjecture. The following result by Garey and Graham [8] could be a starting point for constructing such instances. They analyzed the worst-case performance of the greedy procedure for 2-$\mathcal{UDT}$ and by constructing a family of examples, showed that the procedure's approximation ratio is $\Omega(\log N/\log\log N)$, where N is the number of entities. Another approach to resolve the Erdös' conjecture under the $P \neq NP$ assumption would be to show an $\Omega(\log K \ln N)$ hardness of approximation for the K-$\mathcal{DT}$ problem.

Applications of the $\mathcal{DT}$ problem. Decision trees for entity identification (as defined in this paper) have been used for medical diagnosis (as described earlier), species identification in biology, fault detection etc. [15]. Taxonomists release field guides to help identify species based on their characteristics. These guides are often presented in the form of decision trees labeled by species characteristics. Typically, a field biologist identifies the species of a specimen at hand by referring to such guides (hopefully with as few look-ups as possible). Taxonomists refer to such decision trees as "identification keys" and an article on identification keys can be found in [28]. There are several online interactive guides for species identification: Californian lawn weeds [18], Aquatic macroinvertebrates [20], Snakes of Florida [19], South Australian frogs [2]. In fact, computer programs and algorithms for identification and diagnosis applications have been developed for nearly four decades (e.g., [21, 24, 27]).

Murthy [16] and Moret [15] present excellent surveys on the use of decision trees in such diverse fields as machine learning, pattern recognition, taxonomy, switching theory and boolean logic.

Organization. In Section 2, we present a formal definition of the $\mathcal{DT}$ problem. The black-box and the greedy approximation algorithms are given in Section 3. Section 4 analyzes the greedy algorithm and proves an approximation guarantee. Hardness of approximation results are discussed in Section 6. Section 7 concludes the paper posing some challenging open problems.

2. PROBLEM DEFINITION

Let $\mathcal{D}$ be a relational table having N tuples and m attributes. We call each tuple an *entity*. Let $\mathcal{E}$ and $\mathcal{A}$ denote the set of entities and attributes, respectively. For $x \in \mathcal{E}$ and $a \in \mathcal{A}$, $x.a$ denotes the value of the entity x on the attribute a. For $a \in \mathcal{A}$, $\mathcal{V}_a$ denotes the set of distinct values taken by a in $\mathcal{D}$. Let $K = \max_{a\in\mathcal{A}}\{|\mathcal{V}_a|\}$. Notice that $K \leq N$. We call K the *branching factor* of $\mathcal{D}$.

A *decision tree* T for the table $\mathcal{D}$ is a rooted tree satisfying the following properties. Each internal node u is labeled by an attribute a and has at most K children. Every branch (edge) out of u is labeled by a distinct value from the set $\mathcal{V}_a$. The entities are the leaves of the tree and thus the tree has exactly N leaves. The main requirement is that the tree should identify every entity correctly. Meaning, for any entity x, the following traversal process should correctly lead to x. The process starts at the root node. Let u be the current node and a be the attribute label of u. Take the branch out of u labeled by $x.a$ and move to the corresponding child of u. The requirement is that this traversal process should reach the entity x.

Observe that the values of the attributes are used only for taking the correct branch in the traversal process. So, we can map each value of an attribute to a distinct number from 1 to K and assume that $\mathcal{V}_a$ is a subset of $\{1, 2, \ldots, K\}$. In the rest of the paper, we assume that for any $x \in \mathcal{E}$ and $a \in \mathcal{A}$, $x.a \in \{1, 2, \ldots, K\}$.

As mentioned in the introduction, input to the $\mathcal{DT}$ problem also consists of a probability distribution $\mathcal{P}$ over the set of entities $\mathcal{E}$ specifying the likelihood of the occurrence of each entity. Let T be a decision tree. For an entity x, its path length is defined to be the number of internal nodes in the path from the root to x and denoted by $d_T(x)$. For an entity x, $p_T(x)$ denotes the probability of the occurrence of x as a query. The term $p_T(x)d_T(x)$ is said to be the *weighted path length* of x. The sum of the weighted path lengths of all entities is called the *weighted external path length* of the tree T.

$\mathcal{DT}$ **Problem:** Given a relational table $\mathcal{D}$ and a probability distribution $\mathcal{P}$ over the set of entities, construct a decision tree of minimum cost, where the cost of a tree is its weighted external path length. We assume that the probabilities in $\mathcal{P}$ are input as follows: For each entity e, p_e is input as w_e/L where w_e is the integer weight of the entity e and L is a large integer denominator.

For a positive integer K, the K-$\mathcal{DT}$ problem is a special case of the $\mathcal{DT}$ problem where the input is required to be a table having a branching factor at most K. Notice that in the K-$\mathcal{DT}$ problem, the input is a table whose entries are drawn from the set $\{1, 2, \ldots, K\}$. Given the representation of probabilities, it does not change the problem if we replace the probability p_e of an entity by w_e in the definition of the weighted path length. When convenient, we deal with the weights instead of probabilities and refer to the $\mathcal{DT}$ problem as the *weighted decision tree* problem.

Of particular interest to us is the special case of the $\mathcal{DT}$ problem in which the probability distribution on the set of entities is uniform with $p(e) = 1/N$, $\forall e \in \mathcal{E}$. In this case, all the weights are equal to 1 and we call it as the $\mathcal{UDT}$ problem or the *unweighted decision tree* problem. The cost of a decision tree T is given by $\sum_{e\in\mathcal{E}} d(e)$ and is denoted by $|T|$. Special case of the $\mathcal{UDT}$ problem in which all the input instances have a branching factor of atmost K is known as the K-$\mathcal{UDT}$ problem.

Notations: For a decision tree T of $\mathcal{D}$, we use "$u \in T$" to mean that u is an internal node in T. We denote by $\langle x, y\rangle$, an unordered pair of distinct entities. For brevity, we may refer to decision trees simply as trees.

3. APPROXIMATION ALGORITHMS FOR THE $\mathcal{UDT}$ PROBLEM

It is known that the 2-$\mathcal{UDT}$ problem is NP-hard [14, 9]. In this section, we present two approximation algorithms for the $\mathcal{UDT}$ problem in which the probability distribution is uniform.

3.1 The Black-Box Algorithm

We first present an algorithm which uses the $(1+\ln N)$-approximation algorithm for the 2-$\mathcal{UDT}$ problem by Heeringa and Adler [13] (referred to as the HA-algorithm) as a black box. We encode the given $\mathcal{UDT}$ instance as a 2-$\mathcal{UDT}$ instance and then invoke the HA-algorithm on the encoded instance.

Given an $N \times m$ table $\mathcal{D}$ having a branching factor of K, we construct an $N \times m\lceil\log K\rceil$ binary table $\mathcal{D}_2$ as follows. Each attribute in $\mathcal{D}$ is represented by $\lceil\log K\rceil$

attributes in $\mathcal{D}_2$. The former attribute is called the original attribute and the latter attributes are called as its *derived* attributes. The values appearing in an original attribute are represented in binary in the corresponding derived attributes. Invoke HA-algorithm on the binary table $\mathcal{D}_2$ and let $\mathcal{T}_2$ be the decision tree returned by the algorithm. We obtain a decision tree $\mathcal{T}$ for $\mathcal{D}$ from $\mathcal{T}_2$ by replacing the attributes in its internal nodes with their original attributes in $\mathcal{D}$ and labeling appropriately. Notice that $|\mathcal{T}| \leq |\mathcal{T}_2|$.

Given a tree T for $\mathcal{D}$, we can construct a tree T_2 for $\mathcal{D}_2$ such that $|T_2| \leq \lceil \log K \rceil |T|$. In constructing a decision tree T_2 for the encoded instance $\mathcal{D}_2$, the main task is to take the correct branches of the internal nodes of T using the binary derived attributes. We achieve this by replacing each internal node with a complete binary tree of depth $\lceil \log K \rceil$ using the derived attributes of the original attribute of the internal node. Clearly, $|T_2| \leq \lceil \log K \rceil |T|$. This shows that $|\mathcal{T}_2^*| \leq \lceil \log K \rceil |\mathcal{T}^*|$ where $\mathcal{T}^*$ and $\mathcal{T}_2^*$ are the optimal decision trees for $\mathcal{D}$ and $\mathcal{D}_2$, respectively. Since $|\mathcal{T}_2| \leq (1+\ln N)|\mathcal{T}_2^*|$, the solution $\mathcal{T}$ returned by the black-box algorithm satisfies $|\mathcal{T}| \leq \lceil \log K \rceil (1+\ln N)|\mathcal{T}^*|$.

Theorem 3.1. *The black-box algorithm has an approximation ratio of $\lceil \log K \rceil (1 + \ln N)$ for the $\mathcal{UDT}$ problem where, K is the branching factor of the input table.*

3.2 The Greedy Algorithm

In this section, we present a greedy algorithm for the $\mathcal{UDT}$ problem having an approximation guarantee better than the black-box algorithm. The algorithm is similar in spirit to the Heeringa-Adler algorithm [13] for the 2-$\mathcal{UDT}$ problem. We build on their analysis and develop further combinatorial arguments to establish a connection to Ramsey theory. Using results from Ramsey theory, we then obtain improved approximation ratios.

Given as input an $N \times m$ table $\mathcal{D}$ having branching factor at most K, the greedy algorithm produces a decision tree $\mathcal{T}$ as described below. Let $\mathcal{E}$ and $\mathcal{A}$ denote the set of entities and attributes of $\mathcal{D}$, respectively. The intuition is that any decision tree should distinguish every pair of distinct entities. So, a natural idea is to make the attribute that distinguishes the maximum number of pairs as the root of $\mathcal{T}$, where an attribute a is said to distinguish a pair $\langle x, y\rangle$, if $x.a \neq y.a$. Choosing such an attribute $\widehat{a}$ can be easily done in time $O(mN^2)$. Picking the attribute $\widehat{a}$ as the label for the root node partitions the set $\mathcal{E}$ into disjoint sets $E_1, E_2, \ldots, E_K$, where $E_i = \{x | x.\widehat{a} = i\}$. We recursively apply the same greedy procedure on each of these sets to obtain K decision trees and make these the subtrees of the root node. The greedy procedure is formally specified in Figure 2. We get the output tree $\mathcal{T}$ by calling $\mathcal{T} = \text{Greedy}(\mathcal{E})$.

Theorem 3.2. *The greedy algorithm has an approximation ratio of $(r_K(1 + \ln N))$ for the $\mathcal{UDT}$ problem, where K is the branching factor of the input table. For the K-$\mathcal{UDT}$ problem, the same approximation ratio holds.*

```
Procedure Greedy($E$)
Input: $E \subseteq \mathcal{E}$, a set of entities in $\mathcal{D}$
Output: A decision tree $T$ for the set $E$
Begin
    1. If $|E| = 1$,
           Return a tree with $x \in E$ as a singleton node.
    2. Let $\widehat{a}$ be the attribute that distinguishes the maximum
       number of pairs in $E$:
           $\widehat{a} = \text{argmax}_{a \in \mathcal{A}} |\{(x,y) | x.a \neq y.a\}|$
    3. Create the root node $r$ with $\widehat{a}$ as its attribute label.
    4. For $1 \leq i \leq K$,
           A. Let $E_i = \{x \in E | x.\widehat{a} = i\}$
           B. $T_i = \text{Greedy}(E_i)$
           C. Let $r_i$ be the root of $T_i$. Add $T_i$ to $T$ by adding a
              branch from $r$ to $r_i$ with label $i$.
    5. Return $T$ with $r$ as the root.
End
```

Figure 2: The Greedy Algorithm

4. ANALYSIS OF THE GREEDY ALGORITHM

The analysis is divided into two parts. In the first part, we introduce certain combinatorial objects called tabular partitions and analyze the performance of the greedy algorithm using these objects. In the second part, we relate these objects to Ramsey colorings and complete the proof of Theorem 3.2.

4.1 Analysis Involving Tabular Partitions

Let the input be an $N \times m$ table $\mathcal{D}$ having a branching factor of at most K. Let $\mathcal{E}$ and $\mathcal{A}$ denote the set of all entities and attributes of $\mathcal{D}$, respectively. Let $\mathcal{T}$ and $\mathcal{T}^*$ be the greedy and the optimal decision trees, respectively. In this section, we prove a relationship between $|\mathcal{T}|$ and $|\mathcal{T}^*|$ involving tabular partitions, defined below.

Definition 4.1 (Tabular Partitions). *For an integer $n > 0$, a tabular partition P of n is a sequence $P_1, P_2, \ldots, P_n$ such that P_i is a partition of the set $\{1, 2, \ldots, n\} - \{i\}$. We require that for any distinct $1 \leq i, j \leq n$, if A is the set in P_i containing j and B is the set in P_j containing i, then $A \cap B = \emptyset$. Let the length of a partition P_i denote the number of sets in it. We define the compactness of P as $\texttt{comp}(P) = \max_i(\text{length of } P_i)$, for $1 \leq i \leq n$. We define C_n to be the smallest number such that there exists a tabular partition of n having compactness C_n.*

Theorem 4.2. $|\mathcal{T}| \leq C_K(1 + \ln N)|\mathcal{T}^*|$.

Our main task in this section is to prove the above result. In the next section, we shall show that $C_K \leq r_K$ and obtain Theorem 3.2 by combining the two results. We start with some notations and observations. Let T be any decision tree for $\mathcal{D}$ and u be an internal node of T. We define $E^T(u) \subseteq \mathcal{D}$ to be the set of entities in the subtree of T under u.

Proposition 4.3. *For any decision tree T of $\mathcal{D}$, we have $|T| = \Sigma_{u \in T} |E^T(u)|$.*

Proof. Each entity x contributes a cost equal to its distance from the root. Let us distribute this cost uniformly among the internal nodes on the path from x to the root. Observe that the total cost accumulated at an internal node u is equal to $|E^T(u)|$. Thus, the total cost $|T|$ is equal to $\Sigma_{u \in T} |E^T(u)|$. $\square$

Consider a decision tree T and a pair $\langle x, y\rangle$ of entities. We say that a node $u \in T$ *separates* the pair $\langle x, y\rangle$, if the traversal for both x and y passes through u, but x and y take different branches from u. Formally, u is said to separate [2] $\langle x,y\rangle$, if $x,y \in E^T(u)$ and $x.a \neq y.a$, where a is the attribute label of u. For any pair $\langle x,y\rangle$ of entities, there exists a unique separator in T that separates x and y. We define $\texttt{SEP}(u)$ to be the set of all pairs separated by u. The separators with respect to the greedy tree $\mathcal{T}$ will be important in our analysis. For each pair $\langle x,y\rangle$, we denote by $s_{x,y}$ the separator of $\langle x,y\rangle$ in $\mathcal{T}$ and let $S_{x,y}$ denote $E^{\mathcal{T}}(s_{x,y})$.

From Proposition 4.3, we see that each node $u \in \mathcal{T}$ contributes a cost of $|E^{\mathcal{T}}(u)|$ towards the total cost $|\mathcal{T}|$ and separates the pairs in $\texttt{SEP}(u)$. We distribute the cost $|E^{\mathcal{T}}(u)|$ equally among the pairs in $\texttt{SEP}(u)$. For each pair $\langle x,y\rangle \in \texttt{SEP}(u)$, we define the cost $c_{x,y} = |E^{\mathcal{T}}(u)|/|\texttt{SEP}(u)|$. Since each pair has a unique separator, the costs $c_{x,y}$ are well-defined.

It is easy to see that $|E^{\mathcal{T}}(u)| = \sum_{\langle x,y\rangle\in\texttt{SEP}(u)} c_{x,y}$ and by Proposition 4.3, we have $|\mathcal{T}| = \sum_{\langle x,y\rangle} c_{x,y}$, where the summation is taken over all (unordered) pairs of distinct entities. Notice that each pair $\langle x,y\rangle$ also has a unique separator in $\mathcal{T}^*$. So, we rewrite the above summation by partitioning the set of all pairs according to their separators in $\mathcal{T}^*$ and obtain the following equation:

$$|\mathcal{T}| = \sum_{z\in\mathcal{T}^*} \sum_{\langle x,y\rangle\in\texttt{SEP}(z)} c_{x,y} \qquad (1)$$

For each $z \in \mathcal{T}^*$, we define $\alpha(z)$ to be the the term corresponding to z in the summation given in Equation 1. Clearly, $\alpha(z) = \sum_{\langle x,y\rangle\in\texttt{SEP}(z)} c_{x,y}$. The following lemma gives an upperbound on $\alpha(z)$.

Lemma 4.4. *For any $z \in \mathcal{T}^*$, $\alpha(z) \le C_K(1+\ln|Z|)|Z|$, where $Z = E^{\mathcal{T}^*}(z)$.*

Assuming the correctness of Lemma 4.4, we first prove Theorem 4.2. The lemma is proved later in the section.
Proof of Theorem 4.2: Replacing the inner summation in Equation 1 by $\alpha(z)$ we have

$$|\mathcal{T}| \le C_K(1+\ln N)\sum_{z\in\mathcal{T}^*}|E^{\mathcal{T}^*}(z)| = C_K(1+\ln N)|\mathcal{T}^*|.$$

The first step is obtained by invoking Lemma 4.4 and the fact that $|Z| \le N$. Proposition 4.3 gives us the second step. $\square$

We now proceed to prove Lemma 4.4. Fix any $z \in \mathcal{T}^*$. Let us denote $Z = E^{\mathcal{T}^*}(z)$. Let a_z be the attribute label of z. The node z partitions the set Z into K sets $Z_1, Z_2, \ldots, Z_K$, where $Z_i = \{x \in Z | x.a_z = i\}$. We extend the above notations to sets of values. For any $A \subseteq \{1,2,\ldots,K\}$, define $Z_A = \cup_{i\in A} Z_i$. We have the following upperbound on $c_{x,y}$ (See Appendix for the proof).

Lemma 4.5. *Let $\langle x,y\rangle \in \texttt{SEP}(z)$. Consider disjoint sets $A,B \subseteq \{1,2,\ldots,K\}$ satisfying $y \in Z_A$ and $x \in Z_B$. Then,*

$$c_{x,y} \le \frac{1}{|S_{x,y}\cap Z_A|} + \frac{1}{|S_{x,y}\cap Z_B|}.$$

For each $\langle x,y\rangle$, we shall a choose a suitable pair of disjoint sets A and B and obtain an upperbound on $c_{x,y}$ by invoking Lemma 4.5. We make use of tabular partitions for choosing these sets; the motivation for doing so will become clear in the proof of Lemma 4.8. Let P^* be an optimal tabular partition of K having compactness C_K, given by the sequence $P_1, P_2, \ldots, P_K$. Consider any pair $\langle x,y\rangle \in \texttt{SEP}(z)$. Let $i = x.a_z$ and $j = y.a_z$ so that $x \in Z_i$ and $y \in Z_j$. Let $\widehat{A}$ be the set in the partition P_i that contains j and $\widehat{B}$ be the set in the partition P_j that contains i. Notice that, by the definition of tabular partitions, the sets $\widehat{A}$ and $\widehat{B}$ are disjoint. We invoke Lemma 4.5 with $\widehat{A}$ and $\widehat{B}$ as the required disjoint sets. (Observe that for any i and j, all the pairs in $Z_i \times Z_j$ will make use of the same disjoint sets while invoking the lemma. Thus the sets chosen depend only on the values $x.a_z$ and $y.a_z$). Therefore,

$$c_{x,y} \le \frac{1}{|S_{x,y}\cap Z_{\widehat{A}}|} + \frac{1}{|S_{x,y}\cap Z_{\widehat{B}}|}.$$

We split the above cost into two parts and attribute the first term to x and the second term to y. Define

$$c^x_{x,y} = \frac{1}{|S_{x,y}\cap Z_{\widehat{A}}|} \quad \text{and} \quad c^y_{x,y} = \frac{1}{|S_{x,y}\cap Z_{\widehat{B}}|}.$$

It follows that $c_{x,y} \le c^x_{x,y} + c^y_{x,y}$. For any $x \in Z$, we imagine that x pays a cost $c^x_{x,y}$ to get separated from an entity $y \in Z$. We denote the accumulated cost as $\texttt{Acc}_z(x)$ and define it as

$$\texttt{Acc}_z(x) = \sum_{y:\langle x,y\rangle\in\texttt{SEP}(z)} c^x_{x,y}.$$

Now the lemma given below follows easily.

Lemma 4.6. *For any z, $\alpha(z) \le \Sigma_{x\in Z}\texttt{Acc}_z(x)$.*

Our next task is to obtain an upperbound on $\texttt{Acc}_z(x)$, so that we get a bound on $\alpha(z)$. See Appendix for the proof of the following lemma.

Lemma 4.7. *Let $x \in \mathcal{E}$ be any entity and $Q \subseteq \mathcal{E}$ be any set of entities such that $x \notin Q$. Then,*

$$\sum_{y\in Q}\frac{1}{|S_{x,y}\cap Q|} \le (1+\ln|Q|).$$

Lemma 4.8. *For any $x \in Z$, $\texttt{Acc}_z(x) \le C_K(1+\ln|Z|)$*

Proof. Let $r = x.a_z$ and so $x \in Z_r$. Let $\widetilde{Z} = Z - Z_r$ be the rest of the entities in Z. Notice that $\texttt{Acc}_z(x) = \Sigma_{y\in\widetilde{Z}} c^x_{x,y}$. We perform the above summation by partitioning $\widetilde{Z}$ according to P_r, the r^{th} member of the optimal tabular partition $P^* = P_1, P_2, \ldots, P_K$. Let $P_r = s_1, s_2, \ldots, s_\ell$, where $\ell \le C_K$. For $1 \le i \le \ell$, define $Q_i = \{y \in \widetilde{Z} | y.a_z \in s_i\}$. Thus, $\widetilde{Z} = Q_1 \cup Q_2 \cup \ldots \cup Q_\ell$ and hence,

$$\texttt{Acc}_z(x) = \sum_{1\le i\le\ell}\sum_{y\in Q_i} c^x_{x,y}. \qquad (2)$$

We derive an upperbound for each term in the outer sum using Lemma 4.7. Fix any $1 \le i \le \ell$. Notice that for any $y \in Q_i$, we have $c^x_{x,y} = 1/|S_{x,y}\cap Q_i|$, by definition. Moreover, $x \notin Q_i$. Thus, by applying Lemma 4.7 on Q_i, we get

$$\sum_{y\in Q_i} c^x_{x,y} \le (1+\ln|Q_i|) \le (1+\ln|Z|). \qquad (3)$$

[2]We note that the separator of $\langle x,y\rangle$ is nothing but the least common ancestor of x and y.

We get the lemma by combining Equations 2 and 3, and the fact that $\ell \le C_K$. □

Proof of Lemma 4.4: The result is proved by combining Lemma 4.6 and Lemma 4.8.

$$\begin{aligned}\alpha(z) &\le \sum_{x \in Z} \texttt{Acc}_z(x) \\ &\le \sum_{x \in Z} C_K(1 + \ln|Z|) \\ &= C_K(1 + \ln|Z|)|Z|. \qquad (4)\end{aligned}$$

□

4.2 Tabular Partitions and Ramsey Colorings

In this section, we investigate tabular partitions and relate them to Ramsey colorings. For this purpose, we introduce the notion of directed Ramsey colorings, and show that they are equivalent to tabular partitions. Throughout the discussion, for $n > 0$, let G_n and $\widetilde{G}_n$ denote the complete undirected and the complete directed graph on n vertices, respectively.

Definition 4.9. *Let $n > 0$ be an integer. A directed Ramsey coloring of $\widetilde{G}_n$ is a coloring $\widetilde{\tau}$ of the edges such that for any triplet of distinct vertices x, y and z, if $\widetilde{\tau}(x,y) = \widetilde{\tau}(x,z)$ then $\widetilde{\tau}(y,x) \ne \widetilde{\tau}(y,z)$ (and by symmetry, $\widetilde{\tau}(z,x) \ne \widetilde{\tau}(z,y)$).*

We define $\widetilde{R}_k$ to be the smallest number n such that $\widetilde{G}_n$ cannot be directed Ramsey colored using k colors [3]. The inverse of these numbers will be useful. Define $\widetilde{r}_n$ to be the minimum number of colors required to do a directed Ramsey coloring of $\widetilde{G}_n$.

We claim that for any n, there exists a tabular partition P of compactness k if and only if there exists a directed Ramsey coloring $\widetilde{\tau}$ of $\widetilde{G}_n$ that uses only k colors. A proof sketch follows. Let $P = P_1, P_2, \ldots P_n$. Fix $1 \le x \le n$. Arrange the sets in the partition P_x in an arbitrary manner, say $P_x = s_{x,1}, s_{x,2}, \ldots, s_{x,\ell}$, where $\ell \le k$. The n-1 edges outgoing from the vertex x are colored according to the partition P_x. Meaning, for $1 \le c \le \ell$, for $y \in s_{x,c}$, we set $\widetilde{\tau}(x,y) = c$. For any y and z, if $\widetilde{\tau}(x,y) = \widetilde{\tau}(x,z)$, then it means that y and z belong to the same set in the partition P_x. By the property of tabular partitions, it should be the case that x and z belong to different sets in the partition P_y, implying that $\widetilde{\tau}(y,x) \ne \widetilde{\tau}(y,z)$. We conclude that $\widetilde{\tau}$ is a directed Ramsey coloring and that $\widetilde{\tau}$ uses only k colors. The converse is proved using a similar argument. The claim implies the following proposition.

Theorem 4.10. *For any n, $C_n = \widetilde{r}_n$.*

Let us call an edge-coloring of G_n a Ramsey coloring, if it does not induce any monochromatic triangles. For any n, a Ramsey coloring τ of G_n readily yields a directed Ramsey coloring $\widetilde{\tau}$ of $\widetilde{G}_n$. For each pair of vertices x and y, we set $\widetilde{\tau}(x,y) = \widetilde{\tau}(y,x) = \tau(x,y)$. It can easily be verified that $\widetilde{\tau}$ is indeed a directed Ramsey coloring of $\widetilde{G}_n$. The number of colors used in $\widetilde{\tau}$ is the same as that of τ. Therefore, we have the following proposition.

Proposition 4.11. *For any n, $\widetilde{r}_n \le r_n$.*

[3]Such a number exists, as shown in Theorem 4.13

Proof of Theorem 3.2: The result follows from Theorem 4.2 and 4.10, and Proposition 4.11 □

Let us compare the performance of the black-box and the greedy algorithms. We proved an approximation ratio of $\lceil \log K \rceil (1 + \ln N)$ for the black-box algorithm (Theorem 3.1) and an approximation ratio of $r_K(1 + \ln N)$ for the greedy algorithm (Theorem 3.2). As discussed in the introduction, for any n, $r_n \le 2 + 0.64 \log n$ and so, the approximation ratio of the greedy algorithm is a constant factor improvement over that of the black-box algorithm. Any improvement in the upperbound of r_n would automatically improve the approximation ratio of the greedy algorithm. However, unfortunately, we cannot hope to see too much of an improvement, since it is known that $r_n = \Omega(\log n / \log\log n)$ [26]. Thus, using our approach, we may at best be able to prove that the greedy algorithm has an approximation ratio of $O((\log K / \log\log K) \log N)$. Even obtaining this result seems unlikely in the light of the following conjecture by Erdös(see [17]).

Conjecture 4.12 **(Erdös).** *There exists a constant $\alpha > 0$ such that $R_k \le \alpha^k$, for all k.*

Equivalently, the conjecture says that $r_n = \Omega(\log n)$. Observe that our approximation ratio actually involves $\widetilde{r}_n$, instead of r_n. Thus, we can try to derive a better upper bound on $\widetilde{r}_n$. Unfortunately, we show that $\widetilde{r}_n = \Omega(\log n / \log\log n)$. The claim is implied by the following theorem which can be proved based on an argument similar to the one used to obtain the same bound for R_k.

Theorem 4.13. *For any k, $\widetilde{R}_k \le 1 + k!e$*

Our results on the $\mathcal{DT}$ problem lead to an interesting approach to proving the Erdös' conjecture. The idea is to show that, in terms of worst-case performance factors, the greedy algorithm performs poorly! We observe that a lowerbound of $\Omega(\log K \log N)$ on the approximation ratio for the greedy algorithm would imply the conjecture. More explicitly, we note that the following hypothesis implies the conjecture.

Hypothesis: There exists a constant $\beta > 0$ such that for any K, there exists an instance of the K-$\mathcal{DT}$ problem on which the decision tree $\mathcal{T}$ produced by the greedy algorithm is such that $|\mathcal{T}| \ge \beta \log K \log N |\mathcal{T}^*|$, where $\mathcal{T}^*$ is the optimal decision tree for the instance.

5. WEIGHTED CASE

Let D be the input $N \times m$ table over a set of entities $\mathcal{E}$ and a set of attributes $\mathcal{A}$, having a branching factor of K. Let $w(\cdot)$ be the input weight function that assigns an integer weight $w(x) \ge 1$ to each entity $x \in \mathcal{E}$. Consider a decision tree T of $\mathcal{D}$. For an entity $x \in \mathcal{E}$, let $d_T(x)$ denote the length of path in $\mathcal{T}$ from the root to x. The cost of T with respect to $w(\cdot)$ is defined to be as

$$w(T) = \sum_{x \in \mathcal{E}} w(x) d_T(x).$$

The problem is to construct the optimal decision tree $\mathcal{T}^*$ having the minimum cost with respect to $w(\cdot)$. We generalize the greedy algorithm (Figure 2) to the weighted case.

Weighted Greedy Algorithm: The main step in the greedy algorithm is choosing an attribute that distinguishes the maximum number of pairs. We modify this step so that the weights are taken into account. Namely, we choose the following attribute $\widehat{a}$:

$$\widehat{a} = \text{argmax}_{a\in\mathcal{A}} \sum_{\langle x,y\rangle\in S(a)} w(x)w(y),$$

where $S(a) = \{\langle x,y\rangle | x,y \in \mathcal{E} \text{ and } x.a \neq y.a\}$, is the set of pairs distinguished by the attribute a. We call the above procedure the weighted greedy algorithm.

The following result, which was obtained independently by Gupta and Chaudhary [11], generalizes Theorem 3.2. Let $W = \Sigma_{x\in\mathcal{E}} w(x)$ denote the total weight of the entities. Let $\mathcal{T}$ and $\mathcal{T}^*$ denote the weighted greedy and the optimal trees, under the weight function $w(\cdot)$.

Theorem 5.1. *$w(\mathcal{T}) \leq C_K(1+\ln W)w(\mathcal{T}^*)$, where W is the sum of weights of all the entities.*

We prove the above theorem by adapting the proof of Theorem 3.2. Intuitively, we imagine that each entity x is replicated $w(x)$ times and modify the proof of Theorem 3.2 accordingly. We start by borrowing notations from the above proof and also introduce a few additional notations. Let T be any decision tree for $\mathcal{D}$ and u be an internal node of T. Define $E^T(u) \subseteq \mathcal{D}$ to be the set of entities in the subtree of T under u. Let $\texttt{SEP}(u)$ be the set of all pairs separated by u. For each pair $\langle x,y\rangle$, we denote by $s_{x,y}$ the separator of $\langle x,y\rangle$ in $\mathcal{T}$ and let $S_{x,y}$ denote $E^{\mathcal{T}}(s_{x,y})$.

For a set of entities $X \subseteq \mathcal{E}$, let $w(X)$ denote the total weight of the entities in X, i.e., $w(x) = \Sigma_{x\in X} w(x)$. We also define weights on any set of pairs of entities: for a set of pairs $X \subseteq \mathcal{E}\times\mathcal{E}$, define $w(X) = \Sigma_{\langle x,y\rangle\in X} w(x)w(y)$.

Proposition 4.3 generalizes to the weighted case as follows.

Proposition 5.2. *For any decision tree T of $\mathcal{D}$, $w(T) = \Sigma_{u\in T} w(E^T(u))$.*

For each pair of entities $\langle x,y\rangle$, define a cost $c_{x,y}$ as follows:

$$c_{x,y} = \frac{w(S_{x,y})}{w(\texttt{SEP}(s_{x,y}))}.$$

By Proposition 5.2, we have $w(\mathcal{T}) = \sum_{\langle x,y\rangle} c_{x,y}$, where the summation is taken over all (unordered) pairs of distinct entities. We can rewrite the above summation by partitioning the set of all pairs according to their separators in $\mathcal{T}^*$ and obtain the following equation:

$$w(\mathcal{T}) = \sum_{z\in\mathcal{T}^*} \sum_{\langle x,y\rangle\in\texttt{SEP}(z)} c_{x,y} \tag{5}$$

For each $z \in \mathcal{T}^*$, the inner summation in Equation 5 is defined as the cost $\alpha(z) = \sum_{\langle x,y\rangle\in\texttt{SEP}(z)} c_{x,y}$. Our goal is to derive an upperbound on $\alpha(z)$. Fix any $z \in \mathcal{T}^*$. Let us denote $Z = E^{\mathcal{T}^*}(z)$. Let a_z be the attribute label of z. The node z partitions the set Z into K sets $Z_1, Z_2, \ldots, Z_K$, where $Z_i = \{x \in Z | x.a_z = i\}$. We extend the above notations to sets of values. For any $A \subseteq \{1,2,\ldots,K\}$, define $Z_A = \cup_{i\in A} Z_i$. The following lemma generalizes Lemma 4.5 to the weighted case.

Lemma 5.3. *Let $\langle x,y\rangle \in \texttt{SEP}(z)$. Consider disjoint sets $A,B \subseteq \{1,2,\ldots,K\}$ satisfying $y \in Z_A$ and $x \in Z_B$. Then,*

$$c_{x,y} \leq w(x)w(y)\left[\frac{1}{w(S_{x,y}\cap Z_A)} + \frac{1}{w(S_{x,y}\cap Z_B)}\right].$$

Consider any $\langle x,y\rangle \in \texttt{SEP}(z)$. Let P^* be an optimal tabular partition of K having compactness C_K, given by the sequence $P_1, P_2, \ldots, P_K$. Let $i = x.a_z$ and $j = y.a_z$ so that $x \in Z_i$ and $y \in Z_j$. Let $\widehat{A}$ be the set in the partition P_i that contains j and $\widehat{B}$ be the set in the partition P_j that contains i. Define

$$c^x_{x,y} = \frac{w(x)w(y)}{w(S_{x,y}\cap Z_{\widehat{A}})} \quad\text{and}\quad c^y_{x,y} = \frac{w(x)w(y)}{w(S_{x,y}\cap Z_{\widehat{B}})}.$$

By Lemma 5.3, we have that $c_{x,y} \leq c^x_{x,y} + c^y_{x,y}$. For each entity $x \in E^{\mathcal{T}^*}(z)$, define $\texttt{Acc}_z(x)$ as below:

$$\texttt{Acc}_z(x) = \sum_{y:\langle x,y\rangle\in\texttt{SEP}(z)} c^x_{x,y}.$$

We wish to derive an upperbound on $\texttt{Acc}_z(x)$. The following lemma, which generalizes Lemma 4.7, is useful for this purpose.

Lemma 5.4. *Let $x \in \mathcal{E}$ be any entity and $Q \subseteq \mathcal{E}$ be any set of entities such that $x \notin Q$. Then,*

$$\sum_{y\in Q} \frac{w(y)}{w(S_{x,y}\cap Q)} \leq (1+\ln w(Q)).$$

The following is obtained by generalizing Lemma 4.8.

Lemma 5.5. *For any $x \in Z$, $\texttt{Acc}_z(x) \leq w(x)C_K(1+\ln w(Z))$*

Proof of Theorem 5.1: Consider any $z \in \mathcal{T}^*$ and let $Z = E^{\mathcal{T}^*}(z)$. Then, $\alpha(z) \leq \Sigma_{x\in Z}, \texttt{Acc}_z(x)$. Applying Lemma 5.5 and Proposition 5.2, we get that

$$\alpha(z) \leq C_K(1+\ln w(Z))w(Z). \tag{6}$$

Replacing the inner summation in Equation 5 by $\alpha(z)$ we have

$$\begin{aligned} w(\mathcal{T}) &\leq C_K(1+\ln W)\sum_{z\in\mathcal{T}^*} w(E^{\mathcal{T}^*}(z)) \\ &= C_K(1+\ln W)w(\mathcal{T}^*). \end{aligned} \tag{7}$$

The first step is obtained by invoking Equation 6 and the fact that $w(Z) \leq w(\mathcal{E}) = W$. Proposition 5.2 gives us the second step. □

Theorem 5.1 shows that the approximation ratio of the weighted greedy algorithm is logarithmic in N, when the total weight W is polynomially bounded in N. Unfortunately, when the weights are arbitrary the ratio could be worse. We can overcome the issue by using a standard rounding trick discussed below.

Rounded Greedy Algorithm: Let $\mathcal{D}$ be an input table having a branching factor of K and let w_{in} be the input integer weight function. Let $w^{\max}_{\text{in}} = \max_x w_{\text{in}}(x)$ denote the maximum weight. Define a new weight function $w(\cdot)$ as follows: for any entity $x \in \mathcal{E}$, define

$$w(x) = \left\lceil \frac{w_{\text{in}}(x)N^2}{w^{\max}_{\text{in}}} \right\rceil.$$

Run the weighted greedy algorithm with $w(\cdot)$ as the input weight function and obtain a tree $\mathcal{T}$. Return the tree $\mathcal{T}$.

Let $\mathcal{T}^*$ and $\mathcal{T}^*_{\rm in}$ be the optimal decision trees under the weight functions $w(\cdot)$ and $w_{\rm in}(\cdot)$, respectively. From Theorem 5.1, we have good bound for $w(\mathcal{T})$ with respect to $w(\mathcal{T}^*)$. But, of course, we need to compare $w_{\rm in}(\mathcal{T})$ and $w_{\rm in}(\mathcal{T}^*_{\rm in})$. We do this next.

Theorem 5.6. $w_{\rm in}(\mathcal{T}) \leq 2C_K(1 + 3\ln N)w_{\rm in}(\mathcal{T}^*_{\rm in})$.

PROOF. Let $x \in \mathcal{E}$ be any entity and consider the path from the root to x in the tree $\mathcal{T}^*_{\rm in}$. Notice that each internal node along this path separates at least one entity from x. (Otherwise, $\mathcal{T}^*_{\rm in}$ contains a "dummy" node that does not separate any pairs and hence, can be deleted to obtain a tree of lesser cost). So, the length of the above path is at most N and hence, the following claim is true.

Claim 1: $|\mathcal{T}^*_{\rm in}| \leq N^2$.

We next compare $w_{\rm in}(\mathcal{T}^*_{\rm in})$ and $w(\mathcal{T}^*_{\rm in})$. We have,

$$\begin{aligned}
w(\mathcal{T}^*_{\rm in}) &= \sum_{x\in\mathcal{E}} w(x) d_{\mathcal{T}^*_{\rm in}}(x) \\
&\leq \sum_{x\in\mathcal{E}} \left(\frac{w_{\rm in}(x)N^2}{w^{\max}_{\rm in}} + 1\right) d_{\mathcal{T}^*_{\rm in}}(x) \\
&= \frac{w_{\rm in}(\mathcal{T}^*_{\rm in})N^2}{w^{\max}_{\rm in}} + |\mathcal{T}^*_{\rm in}| \\
&\leq \frac{w_{\rm in}(\mathcal{T}^*_{\rm in})N^2}{w^{\max}_{\rm in}} + N^2 \\
&\leq \frac{2w_{\rm in}(\mathcal{T}^*_{\rm in})N^2}{w^{\max}_{\rm in}}
\end{aligned}$$

The last inequality is obtained by observing the fact that $w_{\rm in}(\mathcal{T}^*_{\rm in}) \geq w^{\max}_{\rm in}$. Thus, we get the following claim.

Claim 2: We have

$$w(\mathcal{T}^*_{\rm in}) \leq \frac{2w_{\rm in}(\mathcal{T}^*_{\rm in})N^2}{w^{\max}_{\rm in}}.$$

Notice that for any entity $x \in \mathcal{E}$, $1 \leq w(x) \leq N^2$ and so the total weight W under the function $w(\cdot)$ satisfies $W \leq N^3$. So, Theorem 5.1 implies the following claim.

Claim 3: $w(\mathcal{T}) \leq C_K(1 + 3\ln N)w(\mathcal{T}^*)$.

We can now compare $w_{\rm in}(\mathcal{T})$ and $w_{\rm in}(\mathcal{T}^*_{\rm in})$. Note that $\mathcal{T}^*$ is the optimal tree under the function $w(\cdot)$ and hence, $w(\mathcal{T}^*) \leq w(\mathcal{T}^*_{\rm in})$. We obtain the lemma by combining the observation with Claims 2 and 3. $\square$

By combining Theorem 5.6, Theorem 4.10 and Proposition 4.11, we get the following result.

Theorem 5.7. *The approximation ratio of the rounded greedy algorithm is at most* $2r_K(1+3\ln N) = O(r_K \log N)$.

6. HARDNESS

In this section, we present the hardness of approximation for the $\mathcal{DT}$ and $\mathcal{UDT}$ problems. Specifically, we show that it is NP-hard to approximate the 2-$\mathcal{DT}$ problem within a ratio of $\Omega(\log N)$. This implies that our results for the 2-$\mathcal{DT}$ problem are asymptotically optimal. While this implies $\Omega(\log N)$ for the $\mathcal{DT}$ problem, improving this to $\Omega(\log K \log N)$ will resolve the Erdös' conjecture discussed earlier. We also present improved hardness of approximation for the $\mathcal{UDT}$ problem.

6.1 Hardness for the 2-$\mathcal{DT}$ problem

In this section, we prove the following theorem:

Theorem 6.1. *It is NP-hard to approximate* 2-$\mathcal{DT}$ *within a factor of* $\Omega(\log N)$, *where* N *is the number of entities in the input.*

PROOF. We prove the result via a reduction from the set cover problem. It is known that approximating set cover within a factor of $\Omega(\log n)$ is NP-hard [23].

Let $(U, \mathcal{S})$ be the input set cover instance, where $U = \{x_1, x_2, \ldots, x_n\}$ is a universe of items and $\mathcal{S}$ is a collection of sets $\{S_1, S_2, \ldots, S_m\}$ such that $S_i \subseteq U$, for each i. Without loss of generality, we can assume that for any pair of distinct items x_i and x_j, there exists a set $S \in \mathcal{S}$ containing exactly one of these two items. (If not, one of these items can be removed from the system.) Construct an instance of weighted 2-$\mathcal{DT}$ problem having $N = n + 1$ entities and m attributes. The set of entities is $\mathcal{E} = \{x_1, x_2, \ldots, x_n\} \cup \{\widehat{x}\}$, where each entity x_i corresponds to the item x_i and $\widehat{x}$ is a special entity. The set of attributes is $\mathcal{A} = \{S_1, S_2, \cdots, S_m\}$, so that each attribute S_i corresponds to the set S_i. The $N \times m$ table $\mathcal{D}$ is given as follows. For each entity x_i and attribute S_j, set $x_i.S_j = 1$, if $x_i \in S_j$ and otherwise, set $x_i.S_j = 0$. For the special entity $\widehat{x}$, set $\widehat{x}.S_j = 0$, for all attributes S_j. For each entity x_i, set the weight $w(x_i) = 1$. As for the special entity $\widehat{x}$, set its weight as $w(\widehat{x}) = N^3$. This completes the construction.

Let T be a decision tree for $\mathcal{D}$. Let C be the set of attributes found along the path from the root to the entity $\widehat{x}$. Recall that the length of the above path is denoted as $d_T(\widehat{x})$. Observe that C is a cover for $(U, \mathcal{S})$. We have $(|C| = d_T(\widehat{x})) \leq w(T)/N^3$. On the other hand, given a cover C, we can construct a decision tree T satisfying the following two properties: (i) the set of attributes along the path from the root to $\widehat{x}$ is exactly the set C so that $|d_T(\widehat{x})| = |C|$; (ii) for every other entity x_i, $d_T(x_i) \leq N$. (The second property is based on the fact that for any table containing N entities, it suffices to test at most N attributes in order to distinguish any entity from the rest). Thus, $w(T) \leq |C|N^3 + N^2$. In particular, $w(T^*) \leq |C^*|N^3 + N^2$, where T^* and C^* are the optimal decision tree and optimal cover, respectively.

Based on the above observations, we can prove the following claim. If there exists an $\alpha(N)$ approximation algorithm for the 2-$\mathcal{DT}$ problem then for any $\epsilon > 0$, we can design an $(1+\epsilon)\alpha(n)$ approximation for the set cover problem. Therefore, the hardness of set cover problem [23] implies the claimed hardness result for the 2-$\mathcal{DT}$ problem. $\square$

6.2 Hardness for the $\mathcal{UDT}$ and the 2-$\mathcal{UDT}$ problems

In this section, we present improved results of hardness of approximation for the $\mathcal{UDT}$ and the 2-$\mathcal{UDT}$ problems. Heeringa and Adler [13] showed a hardness of approximation of $(1 + \epsilon)$, for some $\epsilon > 0$. We show that for any $\epsilon > 0$, it is NP-hard to approximate the $\mathcal{UDT}$ and the 2-$\mathcal{UDT}$ problems within a factor of $(4 - \epsilon)$ and $(2 - \epsilon)$, respectively. Our reductions are from the Minimum Sum Set Cover ($\mathcal{MSSC}$) problem.

The input to the $\mathcal{MSSC}$ problem is a set system: a collection of sets $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ over a universe

$U = \{x_1, x_2, \ldots, x_N\}$ of *items*, where each $S_i \subseteq U$. A solution is an ordering π on the sets in $\mathcal{S}$, with an associated cost defined as follows. Let π be $S'_1, S'_2, \ldots, S'_m$. Each item in S'_1 pays a cost of 1, each item in $S'_2 - S'_1$ pays a cost of 2, and so on. Cost of π is the sum of the costs of all items. Formally, define the costs $c^\pi_x = \text{argmin}_i\{x \in S'_i\}$, for $x \in U$, and $\text{cost}(\pi) = \sum_{x\in U} c^\pi_x$. The $\mathcal{MSSC}$ problem is to find an ordering with the minimum cost. For a constant d, the d-$\mathcal{MSSC}$ problem is the special case of $\mathcal{MSSC}$ in which every set in the set system has at most d elements. Feige et. al [5] proved the following hardness results for these problems.

Theorem 6.2. *[5]*

1. *For any $\epsilon > 0$, it is NP-hard to approximate the $\mathcal{MSSC}$ problem within a ratio of $(4 - \epsilon)$.*
2. *For any $\epsilon > 0$, there exists a constant d such that it is NP-hard to approximate d-$\mathcal{MSSC}$ within a ratio of $(2 - \epsilon)$.*

We next prove the hardness result for the $\mathcal{UDT}$ problem by exhibiting a reduction from $\mathcal{MSSC}$. Given an $\mathcal{MSSC}$ instance $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ over a universe $U = \{x_1, x_2, \ldots, x_N\}$, construct an $N \times m$ table $\mathcal{D}$ as follows. Each item x corresponds to an entity and each set S_i corresponds to an attribute a_i. For $1 \le j \le m$, $1 \le i \le n$, set the entry $x_i.a_j$ as below: if $x_i \in S_j$ then set $x_i.a_j = i$, else set $x_i.a_j = 0$. Observe that any decision tree for $\mathcal{D}$ is *left-deep*: for any internal node u, except the branch labeled 0, every other branch out of u leads to a leaf node.

We claim that given an ordering π of $\mathcal{S}$, we can construct a decision tree $\mathcal{T}$ such that $|\mathcal{T}| = \text{cost}(\pi)$ and vice versa. Let $\pi = S'_1, S'_2, \ldots, S'_m$ and $a'_1, a'_2, \ldots, a'_m$ be the corresponding sequence of attributes. Construct a left-deep tree $\mathcal{T}$, in which the root-node is labeled a'_1 and its 0^{th} child is labeled a'_2 and so on. In general, label the internal node in i^{th} level with a'_i. It can be seen that $\mathcal{T}$ is indeed a decision tree for $\mathcal{D}$ and that $|\mathcal{T}| = \text{cost}(\pi)$. The converse is shown via a similar construction. Given a decision tree $\mathcal{T}$, traverse the tree starting with the root-node and always taking the branches labeled 0. Write down the sequence of sets corresponding to the internal nodes seen in this traversal and let π denote the sequence. Notice that the sets appearing in this sequence cover all elements of U and that $\text{cost}(\pi) = |T|$. (Some sets in $\mathcal{S}$ may not appear in this sequence. To be formally compliant with the definition of solutions, we append the missing sets in an arbitrary order). The claim, in conjunction with Theorem 6.2 (Part 1), implies the following result.

Theorem 6.3. *For any $\epsilon > 0$, it is NP-hard to approximate the $\mathcal{UDT}$ problem within a ratio of $(4 - \epsilon)$.*

The same approach can be used to show the following hardness of approximation for the 2-$\mathcal{DT}$ problem.

Theorem 6.4. *For any $\epsilon > 0$, it is NP-hard to approximate the $2-UDT$ problem within a ratio of $(2 - \epsilon)$.*

7. CONCLUSION

We studied the problem of constructing good decision trees for identification, in the general setup where attributes are multi-valued. We analyzed a natural greedy heuristic and proved an approximation ratio involving Ramsey numbers, and also presented improved hardness of approximation results. There are several interesting open questions. An obvious problem is to bridge the gap between the upper and lower bounds for the approximation ratio of the greedy heuristic. Designing better approximation algorithms and proving improved inapproximability results are also wide-open.

The directed Ramsey numbers $\widetilde{r}_n$ introduced in this paper pose challenging open problems: Is $\widetilde{r}_n = r_n$, for all n? Is $\widetilde{r}_n = O(\log n/\log\log n)$? Proving the second statement in the affirmative would improve the approximation ratio of the greedy procedure by more than a constant factor. If both the statements are shown to be true then the conjecture by Erdös would be disproved! Finally, it would be interesting, if the conjecture can be proved using the approach discussed in Section 4.2.

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APPENDIX

A. OMITTED PROOFS

A.1 Proof of Lemma 4.5

We are given a pair $\langle x,y\rangle \in \texttt{SEP}(z)$. Let $s = s_{x,y}$ be the separator of $\langle x,y\rangle$ in $\mathcal{T}$ and the attribute label of s be a_s. The cost $c_{x,y}$ is given by $|S_{x,y}|/|\texttt{SEP}(s)|$, where $S_{x,y} = E^{\mathcal{T}}(s)$. The greedy algorithm chose the attribute a_s for the node s. Hypothetically, consider choosing the attribute a_z, instead. Let us denote the pairs separated by such a choice as X, i.e., define $X = \{\langle x,y\rangle | x,y \in S_{x,y} \text{ and } x.a_z \neq y.a_z\}$. Notice that the greedy algorithm chose the attribute a_s, instead of a_z, because a_s distinguishes more pairs compared to a_z, meaning, $|\texttt{SEP}(s)| \geq |X|$. It follows that $c_{x,y} \leq |S_{x,y}|/|X|$. Partition $S_{x,y}$ into $S_1, S_2, \ldots, S_K$, where $S_i = \{x \in S_{x,y} | x.a_z = i\}$. Then,

$$|X| = \sum_{1\leq i<j\leq K} |S_i|\cdot|S_j|.$$

We bound the summation by considering only the terms $|S_i|\cdot|S_j|$, where $i \in A$ and $j \in B$. Since $A \cap B = \emptyset$ and $|S_{x,y}| = |S_1| + |S_2| + \cdots + |S_K|$, we have that

$$c_{x,y} \leq \frac{1}{\sum_{i\in A}|S_i|} + \frac{1}{\sum_{j\in B}|S_j|}.$$

Observe that for any $1 \leq i \leq K$, $S_{x,y} \cap Z_i \subseteq S_i$ and hence, $|S_{x,y} \cap Z_i| \leq |S_i|$. Therefore,

$$c_{x,y} \leq \frac{1}{\sum_{i\in A}|S_{x,y}\cap Z_i|} + \frac{1}{\sum_{j\in B}|S_{x,y}\cap Z_j|}.$$

Finally, since the sets $Z_{i'}$ and $Z_{j'}$ are disjoint for any distinct $1 \leq i' \leq j' \leq K$, it follows that the first term equals $1/|S_{x,y} \cap Z_A|$ and the second term equals $1/|S_{x,y} \cap Z_B|$. The lemma is proved. □

A.2 Proof of Lemma 4.7

Let $t = |Q|$. We shall prove the following claim:

$$\sum_{y\in Q} \frac{1}{|S_{x,y}\cap Q|} \leq \sum_{i=1}^{t} \frac{1}{i}.$$

The claim implies the lemma, since it is well known that $\sum_{i=1}^{t}(1/i) \leq (1+\ln t)$, for all t. We prove the claim by applying induction on $|Q|$. For the base case of $|Q| = 1$, let $Q = \{y\}$, where $y \neq x$. Clearly, $y \in S_{x,y}$ and so, $|S_{x,y} \cap Q| = 1$, and the claim follows. Assuming that the claim is true for all sets of size at most $t-1$, we prove it for any set Q of size t. Let y^* be any entity in Q such that for all $y \in Q$, $s_{x,y}$ is a descendent of s_{x,y^*} (a node is considered to be a descendent of itself). If more than one such element exists, pick one arbitrarily. Intuitively, y^* is one among the first batch of entities in Q to get separated from x. The main observation is that $Q \subseteq S_{x,y^*}$ and so, $S_{x,y^*} \cap Q = Q$. Thus, $1/|S_{x,y^*} \cap Q| = 1/|Q| = 1/t$. We apply the induction hypothesis on the set of remaining entities $Q' = Q - y^*$ and infer that

$$\sum_{y\in Q'} \frac{1}{|S_{x,y}\cap Q'|} \leq \sum_{i=1}^{t-1} \frac{1}{i}.$$

Clearly, $Q' \subseteq Q$ and hence, $|S_{x,y} \cap Q'| \leq |S_{x,y} \cap Q|$, So, in the above summation, if we replace the term $|S_{x,y} \cap Q'|$ by $|S_{x,y} \cap Q|$, then the resulting inequality is also true. We conclude that

$$\begin{aligned}\sum_{y\in Q} \frac{1}{|S_{x,y}\cap Q|} &= \frac{1}{|S_{x,y^*}\cap Q|} + \sum_{y\in Q'} \frac{1}{|S_{x,y}\cap Q|} \\ &\leq \frac{1}{t} + \sum_{i=1}^{t-1}\frac{1}{i} = \sum_{i=1}^{t}\frac{1}{i}. \qquad (8)\end{aligned}$$

□