Label optimal regret bounds for online local learning

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Abstract

We resolve an open question from (Christiano, 2014b) posed in COLT'14 regarding the optimal dependency of the regret achievable for online local learning on the size of the label set. In this framework the algorithm is shown a pair of items at each step, chosen from a set of n items. The learner then predicts a label for each item, from a label set of size L and receives a real valued payoff. This is a natural framework which captures many interesting scenarios such as collaborative filtering, online gambling, and online max cut among others. (Christiano, 2014a) designed an efficient online learning algorithm for this problem achieving a regret of $O(\sqrt{nL^3T})$, where T is the number of rounds. Information theoretically, one can achieve a regret of $O(\sqrt{n \log LT})$. One of the main open questions left in this framework concerns closing the above gap.

In this work, we provide a complete answer to the question above via two main results. We show, via a tighter analysis, that the semi-definite programming based algorithm of (Christiano, 2014a), in fact achieves a regret of $O(\sqrt{nLT})$.

Second, we show a matching computational lower bound. Namely, we show that a polynomial time algorithm for online local learning with lower regret would imply a polynomial time algorithm for the planted clique problem which is widely believed to be hard. We prove a similar hardness result under a related conjecture concerning planted dense subgraphs that we put forth. Unlike planted clique, the dense subgraph version does not have quasi-polynomial time algorithms.

Computational lower bounds for online learning are relatively rare, and we hope that the ideas developed in this work will lead to lower bounds for other online learning scenarios as well.

1 Introduction

Online learning is a classic and extensively studied area of machine learning starting from the seminal work of (Littlestone and Warmuth, 1994), (DeSantis et al., 1988) and (Vavock, 1990). In the online learning framework, also known as "prediction from expert advice", the learning algorithm has to predict label information about an item or a set of items at each stage. The learner then earns a real valued payoff which is a function of the predicted labels. The aim of is to achieve a total payoff in T rounds which is as good as the total payoff of the best expert, i.e., the best fixed labeling of the given items. The difference from the best possible payoff is known as the regret of the algorithm.

The weighted majority algorithm (Littlestone and Warmuth, 1994) achieves the optimal regret of $O(\sqrt{T \log N})$ for the above mentioned problem. Here T is the number of rounds and N is the total number of experts. This algorithm, however, is computationally efficient only when the number of experts is small. In many scenarios, one is often competing with a set of exponentially many experts. Hence, not surprisingly, there has been a significant effort in designing polynomial time algorithms with optimal or near optimal regret bounds for various such important problems such as collaborative filtering, online gambling, and online max cut ((Kalai and Vempala, 2005), (Hazan et al., 2012), (Kakade et al., 2009), (Hazan, 2009))

A common aspect of many online learning scenarios mentioned above, is that at each time step, the learner is asked to predict *local* information about items. For instance, in the online max cut problem, the learner has to predict

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whether any two nodes are on the same side of the cut, or on opposite sides. Recently, (Christiano, 2014a) proposed an elegant unifying framework called online local learning to capture such problems.

In this framework, one is given a set of n items, numbered 1 to n. In each round $t \in [T]$, the learner get a pair of items (i_t, j_t) as input, and has to reply with a pair of labels (a_{i_t}, b_{j_t}), where the possible labels are in $[L]$. Then, an adversary picks a payoff function $\mathcal{P}^t : [L]^2 \to [-1, 1]$.

The goal is to compete with the best *fixed* labeling. More precisely, if we denote

$$OPT = max_{\mathbf{l} \in [L]^n} \sum_{t=1}^{T} \mathcal{P}^t \left(l(i_t), l(j_t) \right)$$

and the algorithm achieves a payoff $OPT - r$, the algorithm has regret r.

The main result of (Christiano, 2014a) is that the well known *"Follow-the-regularized-leader"* algorithm with an appropriate regularizer achieves regret $O(\sqrt{nL^3T})$ for the online local learning problem. This, in particular, leads to optimal regret bounds[1] for the online max cut problem. Notice that as mentioned before, one can get the optimal regret of $O(\sqrt{n\ \log L\, T})$ via an inefficient algorithm which runs the weighted majority algorithm over the space of all possible labelings.

One of the main questions left open in this framework was to close the gap between the regret that can be achieved by an efficient algorithm and the information theoretically optimal regret. We close this gap by proving the following results (formal statements appear later).

On the lower bound side, we prove:

Theorem 1 *(Informal). For every $\epsilon > 0$, if there exists an algorithm for online local learning achieving regret $O(\sqrt{nL^{1-\beta(\epsilon)}T})$, and running in time polynomial in n, L, T, then in polynomial time, one can distinguish an instance of a random graph $G(n, 1/2)$ from an instance of $G(n, 1/2)$ with a randomly planted clique of size $n^{1/2-\epsilon}$. Here, $\beta(\epsilon)$ is a function such that $lim_{\epsilon \to 0} \beta(\epsilon) = 0$.*

We also prove a similar lower bound under a more robust conjecture concerning planting dense subgraphs which we introduce, which has no known quasipolynomial time algorithms, unlike planted clique. We show:

Theorem 2 *(Informal). For every $\epsilon, \epsilon' > 0$, if there exists an algorithm for online local learning achieving regret $O(\sqrt{nL^{1-\beta(\epsilon,\epsilon')}T})$, and running in time polynomial in n, L, T, then in polynomial time, one can distinguish between an instance of $G(n, p)$ and an instance of $G(n, p)$ with a randomly planted instance of $G(k, q)$. Here, k, q depend on ϵ, ϵ', and $\beta(\epsilon, \epsilon')$ is a function such that $lim_{\epsilon,\epsilon' \to 0} \beta(\epsilon, \epsilon') = 0$.*

We match the above lower bounds with the following theorem:

Theorem 3 *(Informal). For the online local learning problem, follow the regularized leader with an appropriate regularizer achieves regret $O(\sqrt{nLT})$.*

1.1 Techniques

We obtain the above mentioned upper bound on the regret by showing that "Follow-the-regularized-leader" using the same regularizer as (Christiano, 2014a) achieves the regret bound we are claiming. Our analysis is completely different from the one in (Christiano, 2014a). There, the main idea is that one can express the entropy of a multivariate Gaussian in terms of the logarithm of the determinant of its covariance matrix, and that two multivariate Gaussian distributions that differ by a small amount in their covariance matrices cannot be too far in total variation distance as well.

The main reason for this approach in (Christiano, 2014a) is that the Hessian of the log-determinantal regularizer is not diagonal, so it's difficult to argue about it's inverse. We use the special structure of the regularizer to get explicit expressions for the inverse, which allows us then to use more standard tools from convex geometry for analyzing "Follow-the-regularized-leader". To do this we use some identities from matrix calculus, which we think might be useful in other machine learning applications, where one needs to perform regularized optimization over polytopes of pseudo-moments.

[1] Up to constant factors

Our lower bounds are based on two conjectures about detecting planted dense structures inside random graphs. The first one is planted clique, which states that detecting planted cliques of sufficiently small size in an Erdös-Rényi graph cannot be done in polynomial time. We introduce a more robust version of this conjecture, planted dense subgraph, which concerns detecting planted dense Erdös-Rényi graphs inside sparser ones. While the reductions are similar in both cases, the state of the art algorithms for this detection problem are much worse - planted clique can be solved in $n^{O(\log n)}$ time, while we only know how to solve planted dense subgraph in time $O(2^{n^\epsilon})$ in certain regimes. This is an indicator that this problem is likely harder than planted clique, and gives even stronger evidence for the hardness of achieving low regret.

The lower bounds we show are robust: namely if planted clique is hard when the planted clique is of size $n^{\frac{1}{2}-\epsilon}$, we get a lower bound on the regret $O(\sqrt{nL^{1-\beta(\epsilon)}T})$, such that $\lim_{\epsilon\to 0}\beta(\epsilon)=0$. The situation is similar in the case of planted dense subgraph. In both cases, we use the online learner as an *estimator* of the size of the largest clique or dense subgraph in a graph, and the regret as the *rate* of error in this estimator. We show that if the *rate* is low enough, then one can distinguish between a planted and a non-planted case. See next section for further details.

1.2 The planted dense subgraph and planted clique problems

We will review the planted clique conjecture, and describe the dense subgraph conjecture, upon which we will be basing our lower bounds.

1.2.1 Planted clique

In the planted clique problem, one is given a graph sampled from one of two possible random ensembles: an Erdös-Rényi random graph $G(n, 1/2)$, or an Erdös-Rényi random graph $G(n, 1/2)$ along with a clique of size k placed between k randomly chosen vertices in the graph. (The usual notation for this random ensemble is $G(n, 1/2, k)$.) The task is to distinguish whether one is presented with a graph from the $G(n, 1/2)$ ensemble or the $G(n, 1/2, k)$ ensemble.

Previous sequences of work (Feldman et al., 2013), (Meka et al., 2015), show that wide classes of natural polynomial time algorithms cannot efficiently distinguish between these two cases when the size of the planted clique is $n^{\frac{1}{2}-\epsilon}$, and it is conjectured that in fact there is no polynomial time algorithm for this task. More precisely, the conjecture is the following:

Conjecture 1. *Suppose that an algorithm $\mathcal{A}$ receives as input a graph G, which is either sampled from the ensemble $G(n, 1/2)$ or $G(n, 1/2, n^{\frac{1}{2}-\epsilon})$, $\epsilon = \Omega(1)$. Then, no $\mathcal{A}$ which runs in polynomial time can decide, with probability $\frac{4}{5}$[2], which ensemble the input was sampled from.*

1.2.2 Planted dense subgraph

The planted dense subgraph problem will be a natural generalization of planted clique, where one again wants to distinguish between a random and a planted instance. In the planted case, we plant a denser graph inside a sparser one. Formally, let $G(n, p, k, q)$ be a random graph ensemble generated in the following manner. First, one picks a random subset S of k vertices. Then, for all pairs of vertices inside S, one connects them with an edge independently with probability q. For all other pairs of vertices, we connect them independently with probability p.

The sizes and densities of the planted and ambient graph in which we will be interested is $p = n^{-\alpha}, k = n^{\frac{1}{2}-\epsilon'}, q = k^{-\alpha-\epsilon}$, for $\alpha \leq \frac{1}{2}$. The main reason this scenario is interesting is that unlike planted clique, we do not know of quasi-polynomial time algorithms for it.

To be formal, we conjecture the following:

Conjecture 2. *Suppose that an algorithm $\mathcal{A}$ receives as input a graph G, which is either sampled from the ensemble $G(n, p)$ or $G(n, p, k, q)$, where $k = n^{\frac{1}{2}-\epsilon'}$ for $\epsilon' = \Omega(1)$ and $k = n^{\Omega(1)}$; $p = n^{-\alpha}$ for $\alpha = \Omega(1), \alpha \leq \frac{1}{2}$; $q = k^{-\alpha-\epsilon}$ for $\epsilon = \Omega(1)$; and $p = o(q)$. Then, no $\mathcal{A}$ which runs in polynomial time can decide with probability $\frac{4}{5}$ which ensemble the input was sampled from.*

[2] The constant is arbitrary. One could make the conjecture for any constant bounded away from $\frac{1}{2}$

There are few ways to justify this conjecture. First, the current best known algorithm for this distinguishing problem from (Bhaskara et al., 2010) runs in time $n^{k^{\Theta(\epsilon)}}$. This bound gives a running time of $2^{n^{\Theta(\epsilon)}}$ since k is polynomial in n, which is significantly worse than quasi-polynomial. Second, it's possible to show (Bhaskara et al., 2010) that spectral methods to not work in this regime. It's also easy to check that simple algorithms like outputting the vertices with highest degree do not work either - since the variance of the degree in the sparser ambient graph dominates the degrees in the denser planted graph. Finally, similar conjectures to this have already been proposed in various contexts in theoretical computer science. ((Arora et al., 2010), (Applebaum et al., 2010))

The fact that state of the art algorithms have a much worse running time for this problem in comparison to planted clique is our motivation for putting forth this conjecture. Namely, our reduction of planted clique/planted dense subgraph to online local learning will produce an online learning instance in which the number of items n', the number of rounds T and the label set size L are all polynomial in the size of the input graph. Furthermore, the time to produce the inputs for the learning algorithm will be polynomial as well. Therefore, if $N = \max(T, L, n)$, and we have an algorithm of running time $f(N)$ for online local learning, we get an algorithm for planted clique/planted dense subgraph of running time $\max\left(f\left(poly\left(n\right)\right), poly\left(n\right)\right)$.

This means for instance, if our algorithm for online local learning has running time $f(N) = N^{o(\log N)}$, we would get via our reduction an algorithm for planted clique with running time $n^{o(\log n)}$. A similar statement holds in the planted dense subgraph case. If our algorithm for online local learning has running time even $f(N) = 2^{N^{o(1)}}$, we would get via the reduction an algorithm better than the state of the art for planted dense subgraph.

2 Computational lower bounds on achievable regret

We will proceed with the lower bound first. Recall that we want to show that achieving regret lower than $O(\sqrt{n\, o(L)\, T})$ is impossible.

The overall strategy will be as follows. We will produce an online learning instance from our input graph. In the planted case, there will be a fixed labeling which achieves a large payoff b_p, and in the random case, we'll show that any algorithm (efficient or not) can achieve at most some small payoff b_r. The reduction will ensure that if we can get a sufficiently low regret r in polynomial time, we will get a payoff of at least $b_p - r$ in the planted case, such that $b_p - r \gg b_r$. Then to distinguish between the planted and random case, we simply declare *planted* if the regret is low enough, and *random* otherwise.

For both reductions, we will show a "robust" version of the bound first. For instance, for planted clique, we will show a lower bound of $O(\sqrt{nL^{1-\beta(\epsilon)}T})$ if planted clique is hard when the size of the planted portion is $n^{1/2-\epsilon}$, for some function $\beta(\epsilon)$. Then, we will show that in the limit $\epsilon \to 0$, the lower bound becomes $O(\sqrt{n\, o(L)\, T})$. The details of the reduction follow.

2.1 Planted clique-based hardness

Let us proceed to the planted clique-based lower bound first. We provide a proof sketch of the main result here, and the full proof is included in Appendix A. We will show:

Theorem 1. *Let $\epsilon = \Omega(1)$. If regret $\sqrt{nL^{\beta}T}$ for*

$$\beta = \left(1 - \omega\left(\frac{1}{\log n}\right)\right)\left(\frac{1}{\frac{1}{2} + \epsilon}\right) - 1$$

is achievable in time polynomial in n, L, T, then one can distinguish between $G(n, 1/2)$ and $G\left(n, 1/2, n^{1/2-\epsilon}\right)$ with probability $\frac{4}{5}$[3] in polynomial time.

Proof. We produce an instance for the online local learning problem, given an instance of the planted clique problem with size of the planted clique k in the following way.

[3] Again, the choice of $\frac{4}{5}$ is arbitary

We will randomly partition the graph we are given into $n' = n/l$ clusters, each containing l vertices, where $l = 10\frac{n}{k}$. We associate each vertex with a unique label in $\{1, .., l\}$. We then use this as an instance for the online learning problem as follows. We run it for $T = \binom{n'}{2}$ steps. We query all pairs all pairs of clusters (C_1, C_2) in an arbitrary order. The algorithm responds with some labeling for the clusters (l_i, l_j), and the payoff is 1 if the vertex for l_i in C_1 has an edge to the vertex for l_j in C_2. Otherwise, the payoff is 0.

Let's assume the original graph was from $G(n, 1/2)$. Then, we claim that any algorithm (regardless if efficient or not) will get a payoff of at most $\frac{T}{2} + 5\frac{\sqrt{T}}{2}$ with probability at least $\frac{4}{5}$.

The probability above is with respect to the randomness in generating the graph from $G(n, 1/2)$, the partitioning of the vertices, and any randomness in the algorithm - where the randomness is generated exactly in that order. Since the partitioning is done independently of the graph, we can also just fix one particular partitioning of the vertices, then randomly generate the edges of the graph, and finally run the online learning algorithm on this graph. Finally, by the principle of deferred decisions, we can interleave the generation of the edges and the running of the algorithm, i.e. we can assume the graph is being generated as the online learning game progresses.

More precisely, consider the following two random processes. In the first one, we will assume that we generate a graph from $G(n, 1/2)$, and at time step t, for the pair of clusters (C_1^t, C_2^t) queried at that time step, we reveal the payoffs (equivalently edges) to the algorithm. The distribution of the answers of the algorithm is the same in both.

In the second random process, if $\mathcal{P}_t$ is a random variable denoting the payoff at time t, clearly $\mathcal{P}_t$ is just a Bernoulli $0-1$ variable, which is 1 with probability $\frac{1}{2}$, and the payoffs at different rounds are independent. Hence, by Chernoff,

$$\Pr\left[\sum_{t=1}^{T} \mathcal{P}_t \geq \frac{T}{2} + 5\frac{\sqrt{T}}{2}\right] \leq e^{-50}$$

So, in particular, with probability $\geq \frac{4}{5}$, any algorithm gets payoff of $\frac{1}{2}\binom{k/10}{2} + o(k^2)$.

Let's proceed to the planted case. We claim that with probability at least $\frac{4}{5}$, there is a fixed labeling with payoff at least $\binom{2k/25}{2}$.

Let $\mathcal{I}_i$ be an indicator random variable for the event that no vertex from the planted clique belongs to cluster i. The partitioning is done independently of the graph, so

$$\Pr[\mathcal{I}_i = 1] = \left(1 - \frac{10}{k}\right)^k \leq e^{-\frac{10}{k}k} \leq e^{-10}$$

Hence, if $\mathcal{I}$ is a random variable for the total number of clusters which contain no vertices from the planted clique, we know that

$$\mathbb{E}[\mathcal{I}] = \sum_{i=1}^{n'} \mathbb{E}[\mathcal{I}_i] \leq \frac{k}{10} e^{-10}$$

By Markov's inequality,

$$\Pr\left[\mathcal{I} \geq \frac{k}{50}\right] \leq 5e^{-10} \leq \frac{1}{5}$$

So, with probability at least $\frac{4}{5}$, the number of clusters with at least one vertex from the planted clique is at least $\frac{k}{10} - \frac{k}{50} = \frac{2k}{25}$. In this case, the labeling where we label each of the clusters with a vertex from the planted clique gets a payoff of at least $\binom{2k/25}{2}$.

In this new online learning instance we constructed, the number of vertices is n', the number of rounds is T, and the label size is l. Let's suppose we can achieve regret of $\sqrt{n' l^\beta T}$. Whenever the original graph was a planted instance, the algorithm would get a payoff of at least $\binom{2k/25}{2} - \sqrt{n' l^\beta T}$ with probability at least $\frac{4}{5}$, whereas in a random instance, we'd get a payoff at most $\frac{1}{2}\binom{k/10}{2} + o(k^2)$ with probability at least $\frac{4}{5}$.

It's easy to check that $\binom{2k/25}{2} \geq \left(1 + \frac{1}{100}\right) \frac{1}{2} \binom{k/10}{2}$. Hence, if $\sqrt{n' l^{\beta} T} = o(k^2)$, we can distinguish between a planted and random instance with probability $\frac{4}{5}$. We will show exactly that.

First we claim that

$$l^{\beta} = o\left(\frac{n}{l}\right) \tag{1}$$

Since $l = 10\frac{n}{k} = 10n^{\frac{1}{2}+\epsilon}$, after rearranging terms, 1 is equivalent to

$$n^{(\beta+1)\left(\frac{1}{2}+\epsilon\right)} = o(n)$$

Notice that $n^{\omega(\frac{1}{\log n})} = \omega(1)$, so for the above it is sufficient that

$$(\beta + 1)\left(\frac{1}{2} + \epsilon\right) = 1 - \omega(\frac{1}{\log n})$$

But since

$$\beta = \left(1 - \omega\left(\frac{1}{\log n}\right)\right)\left(\frac{1}{\frac{1}{2} + \epsilon}\right) - 1$$

the above is clearly satisfied.

Hence,

$$\sqrt{n' l^{\beta} T} = \sqrt{\frac{n}{l} l^{\beta} \binom{\frac{n}{l}}{2}} = o\left(\sqrt{l^{\beta}\left(\frac{n}{l}\right)^3}\right) = o\left(\left(\frac{n}{l}\right)^2\right) = o(k^2)$$

which finishes the proof. □

This quite easily will give the result that assuming Conjecture 1, achieving regret $\sqrt{nL^{1-\delta}T}$, for any $\delta = \Omega(1)$ is hard. More precisely:

Corollary 1. *Let $\epsilon = \Omega(1)$. If we can achieve regret $\sqrt{nL^{1-\epsilon}T}$ in time polynomial in n, L, T, we can distinguishing between $G(n, 1/2)$ and $G(n, 1/2, n^{1/2-\frac{\epsilon}{6}})$ with probability $\frac{4}{5}$ in polynomial time. In particular, if Conjecture 1 is true, no polynomial time algorithm can achieve regret $\sqrt{nL^{1-\delta}T}$, for any $\delta = \Omega(1)$.*

Proof. For ease of notation, let's call $\tilde{\epsilon} := \frac{\epsilon}{6}$. Since $\tilde{\epsilon} = \omega(\frac{1}{\log n})$, directly applying Theorem 1, to distinguish between $G(n, 1/2)$ and $G(n, 1/2, n^{1/2-\tilde{\epsilon}})$, it's sufficient to achieve regret $\sqrt{nL^{\beta}T}$, for $\beta = (1 - \tilde{\epsilon})\frac{2}{1 + 2\tilde{\epsilon}} - 1$. Since $\frac{2}{1+2\tilde{\epsilon}} = 2 - \frac{4\tilde{\epsilon}}{1+2\tilde{\epsilon}}$

$$(1 - \tilde{\epsilon})\frac{2}{1 + 2\tilde{\epsilon}} = (1 - \tilde{\epsilon})\left(2 - \frac{4\tilde{\epsilon}}{1 + 2\tilde{\epsilon}}\right) =$$

$$2 - (2 + \frac{4}{1 + 2\tilde{\epsilon}})\tilde{\epsilon} + \frac{2\tilde{\epsilon}^2}{1 + 2\tilde{\epsilon}} \geq 2 - 6\tilde{\epsilon} = 2 - \epsilon$$

Hence, if we can achieve regret $\sqrt{nL^{1-\epsilon}T}$, we can distinguish between $G(n, 1/2)$ and $G(n, 1/2, n^{\frac{1}{2}-\frac{\epsilon}{6}})$, as we needed.

□

We note that a stronger form of Conjecture 1 is consistent with our current knowledge of planted clique. In particular, we can strengthten the claim to allow any $k = o(\sqrt{n})$, or alternatively $k = n^{\frac{1}{2}-\epsilon}$, for any $\epsilon = \omega(\frac{1}{\log n})$. In this case, Corollary 1 will imply that achieving regret $\sqrt{n\, o(L)T}$ is impossible in polynomial time.

2.2 Planted dense graph hardness

We next move on to the planted dense graph based hardness. The proofs in this section are essentially a generalization of the planted clique hardness, so are relegated to Appendix B. We formally show:

Theorem 2. *Let ϵ, α, k satisfy the conditions of Conjecture 2. If regret $\sqrt{nL^{\beta}T}$ for*

$$\beta = 2\frac{\frac{1}{2} - \left(\frac{1}{2} - \epsilon'\right)(\alpha + \epsilon) - \omega\left(\frac{1}{\log n}\right)}{\frac{1}{2} + \epsilon'} - 1$$

is achievable in time polynomial in n, L, T, then one can distinguish between $G(n, p_s)$ and $G(n, p_s, k, p_d)$, where $p_s = n^{-\alpha}, k = n^{\frac{1}{2} - \epsilon'}, p_d = k^{-\alpha - \epsilon}$ with probability $\frac{4}{5}$ in polynomial time.

And again as before, assuming Conjecture 2, achieving regret $\sqrt{nL^{1-\delta}T}$, for any $\delta = \Omega(1)$ is hard. More precisely:

Corollary 2. *Let $\epsilon', \alpha, \epsilon = \Omega(1)$ and $\alpha \geq \epsilon$. If we can achieve regret $\sqrt{nL^{1-\epsilon'-\alpha-\epsilon}T}$ in time polynomial in n, L, T, we can distinguish between $G(n, p_s)$ and $G(n, p_s, k, p_d)$ in polynomial time with probability $\frac{4}{5}$, where $p_s = n^{-\frac{\alpha}{8}}, k = n^{\frac{1}{2} - \frac{\epsilon'}{4}}, p_d = k^{-\frac{\alpha}{8} - \frac{\epsilon}{8}}$. In particular, if Conjecture 2 is true, no polynomial time algorithm can achieve regret $\sqrt{nL^{1-\delta}T}$, for any $\delta = \Omega(1)$.*

Similarly, a stronger form of Conjecture 2 is plausible given our current knowledge. We can allow $\alpha = \omega(\frac{1}{\log n})$, $\epsilon = \omega(\frac{1}{\log k})$, and $k = o(\sqrt{n})$. (These constraints are necessary in order to make sure that $n^{-\alpha} = o(1)$, and $k^{-\epsilon} = o(1)$, since unlike planted clique, we are thinking of p and q as asymptotic quantities, so we want to ensure that $k^{-\alpha-\epsilon} = o(k^{-\alpha})$, and $n^{-\alpha} = o(1)$.) In this case, Corollary 2 will imply that achieving regret $\sqrt{n\, o(L)T}$ is impossible in polynomial time.

3 Improved regret bound analysis of log-determinantal regularizer

We now move to the other result in our paper: matching the lower bound from the previous section. We show that "Follow-the-regularized-leader" with the log-determinant-based regularizer from (Christiano, 2014a) achieves regret $O(\sqrt{nLT})$.

We will follow the (Hazan, 2009) framework for online convex optimization. The scenario is as follows: at each round t, the player chooses a point $\vec{x}_t \in \mathcal{K}$, where $\mathcal{K}$ is some convex body. A linear payoff function is revealed, and the player receives a payoff $\vec{\mathcal{P}}_t \cdot \vec{x}_t$, for some vector $\vec{\mathcal{P}}_t$. The goal is to compete with the "best decision in hindsight", i.e. to maximize

$$\inf_{\vec{\mathcal{P}}_1, \vec{\mathcal{P}}_2, \ldots, \vec{\mathcal{P}}_T} \{\mathbb{E}[\sum_{i=1}^{T} \vec{\mathcal{P}}_i \cdot \vec{x}_i] - \max_{\vec{x} \in \mathcal{K}} \sum_{i=1}^{T} \vec{\mathcal{P}}_i \cdot \vec{x}\}$$

where the expectation is over the randomness of the algorithm.

Then, "Follow-the-regularized-leader", with a convex regularizer $\mathcal{R}(\vec{x})$, is the following algorithm:

Algorithm 1: Follow-the-regularized-leader

$\vec{x}_1 = argmax_{\vec{x}} \mathcal{R}(\vec{x})$;
for $t \leftarrow 1$ **to** T **do**
  Predict $\vec{x}_t$;
  Observe the payoff function $\vec{\mathcal{P}}_t$;
  Update $\vec{x}_{t+1} = argmax_{\vec{x} \in K} \left[\sum_{s=1}^{t} \vec{\mathcal{P}}_s \cdot x - \mathcal{R}(\vec{x})\right]$;

The main theorem in (Hazan, 2009) is:

Theorem. *(Hazan, 2009) "Follow-the-regularized-leader", with a convex regularizer $\mathcal{R}(x)$ and an appropriate choice of ν, achieves regret $O(\sqrt{D\gamma T})$, where*

$$D = \max_{\vec{x}\in\mathcal{K}} |\mathcal{R}(\vec{x})| \ , \ \gamma = \max_{\vec{x}\in\mathcal{K},\vec{\mathcal{P}}_t} \vec{\mathcal{P}}_t^{\intercal}[\nabla^2\mathcal{R}(\vec{x})]^{-1}\vec{\mathcal{P}}_t$$

Since we are following the same approach as in (Christiano, 2014a), for us the polytope $\mathcal{K}$ will be the convex polytope of pseudo-moments, i.e. positive semidefinite matrices $M_{(i,a),(j,b)}$ where $1 \leq i,j \leq n$, $1 \leq a,b \leq L$, such:

- $\forall i,a,j,b$, $1 \geq M_{(i,a),(j,b)} \geq 0$
- $\forall i,j$, $\sum_{a,b} M_{(i,a),(j,b)} = 1$.

Then, $\vec{\mathcal{P}}_t \in [-1,1]^{(nL)^2}$, indexed by all pairs $((i,a),(j,b))$. Furthermore, for any t, there are nonzeros in $\vec{\mathcal{P}}_t$ only over a single pair (i_t,j_t) (the edge that round is played on), and in that case $\vec{\mathcal{P}}_t((i_t,a),(j_t,b))$ is the payoff of playing label a on the i_t vertex, and label b on the j_t vertex. The payoff at round t would be simply

$$\sum_{a,b} \vec{\mathcal{P}}_t((i_t,a),(j_t,b)) \cdot M_{(i_t,a),(j_t,b)}$$

.

The regularizer we use is $\mathcal{R}(M) = \log\det(I + LM)$. In (Christiano, 2014a), it is shown that the diameter parameter D is at most nL, however an additional L^2 factor in the analysis of the γ parameter is lost. (While not quite written in these terms there, the argument in the paper can be very easily cast this way.) Here, we improve that analysis to show that in fact $\gamma \leq 4$.

So, we will simply prove:

Theorem 3. *For online local learning, "follow-the-regularized-leader" with a regularizer $\mathcal{R}(\vec{x}) = \log\det(I + LM)$ achieves regret $O(\sqrt{D\gamma T})$, where*

$$D = \max_{\vec{x}\in\mathcal{K}} |\mathcal{R}(\vec{x})| \leq nL \ , \ \gamma = \max_{\vec{x}\in\mathcal{K},\vec{\mathcal{P}}_t} \vec{\mathcal{P}}_t^{\intercal}[\nabla^2\mathcal{R}(\vec{x})]^{-1}\vec{\mathcal{P}}_t \leq 4$$

3.1 Calculating the inverse Hessian of the regularizer

We'll prove the following lemma first:

Lemma 1. *If $\mathcal{R}(M) = \log\det(I + LM)$, then:*

$$(\nabla^2\mathcal{R}(M))^{-1}_{((i,a),(j,b)),((i',a'),(j',b'))} =$$

$$\frac{1}{L^2}\left(\delta\left((i',a'),(j,b)\right) + L\cdot M\left((i',a'),(j,b)\right)\right)\left(\delta\left((i,a),(j',b')\right) + L\cdot M\left((i,a),(j',b')\right)\right)$$

Proof. Let's proceed stepwise. First, let's calculate the gradient. For this, the following theorem from matrix calculus is very useful (where adj stands for the adjugate):

Theorem. *Jacobi's Formula (Magnus and Neudecker, 1995):*

$$\frac{\partial\det(B)}{\partial B_{i,j}} = adj(B)^{\intercal}_{i,j} = adj(B)_{j,i} = \det(B)B^{-1}_{j,i}$$

With this in mind, the gradient is a simple matter of applying the chain rule. To keep the notation clean, let $w = (i, a), x = (j, b)$, and calculate the gradient of $R(M)$ with respect to $M_{w,x}$. We get:

$$\frac{\partial \mathcal{R}(M)}{\partial M_{w,x}} = \frac{1}{\det(I + L \cdot M)} \frac{\partial \det(I + L \cdot M)}{\partial M_{w,x}} = (I + L \cdot M)^{-1}_{x,w} \frac{\partial (I + L \cdot M)_{w,x}}{\partial M_{w,x}}$$
$$= L(I + L \cdot M)^{-1}_{x,w}$$

Again, to keep the notation lighter, let $y = (i', a'), z = (j', b')$. We will use a little bit of matrix calculus to show:

Lemma 2. $\frac{\partial (I+L\cdot M)^{-1}_{x,w}}{\partial M_{y,z}} = -L(I + L \cdot M)^{-1}_{x,y}(I + L \cdot M)^{-1}_{w,z}$

Proof. Let's denote by $\frac{\partial X}{\partial t}$ the matrix with entries $\frac{\partial X_{i,j}}{\partial t}$. Then, we claim the following is true: $\dfrac{\partial (XY)}{\partial t} = \dfrac{\partial X}{\partial t} Y + X \dfrac{\partial Y}{\partial t}$. This is not hard to check: it's just due to the fact that in the matrix product XY, the entry $(XY)_{i,j}$ is a sum of terms which multiplications of two entries in X and Y. An application of the chain rule gives the above quite easily.

Then, we use the following trick: $BB^{-1} = I$, so by the above observation, $\frac{\partial B}{\partial t} B^{-1} + B \frac{\partial B^{-1}}{\partial t} = 0$. Hence, $\frac{\partial B^{-1}}{\partial t} = -B^{-1} \frac{\partial B}{\partial t} B^{-1}$. Let's apply this observation to $B = (I + L \cdot M)$ and $t = M_{y,z}$

$$\frac{\partial (I + L \cdot M)^{-1}_{x,w}}{\partial M_{y,z}} = -((I + L \cdot M)^{-1} \frac{\partial (I + L \cdot M)}{\partial M_{y,z}} (I + L \cdot M)^{-1})_{x,w} \quad (2)$$
$$= -\sum_{p,q} (I + L \cdot M)^{-1}_{x,p} \frac{\partial (I + L \cdot M)_{p,q}}{\partial M_{y,z}} (I + L \cdot M)^{-1}_{q,w} \quad (3)$$

Now, the term $\frac{\partial (I+L\cdot M)_{p,q}}{\partial M_{y,z}}$ is non-zero only if $p = y, q = z$, in which case it is equal to k. Hence, we get:

$$(3) = -L(I + L \cdot M)^{-1}_{x,y}(I + L \cdot M)^{-1}_{w,z}$$

as needed.

□

With this in mind, the Hessian is obvious:

$$\frac{\partial^2 \mathcal{R}(M)}{\partial M_{w,x} \partial M_{y,z}} = \frac{\partial}{\partial M_{y,z}} L(I + L \cdot M)^{-1}_{x,w} = -L^2(I + L \cdot M)^{-1}_{x,y}(I + L \cdot M)_{w,z}$$

Let's call the Hessian matrix $H_{(w,x),(y,z)}$. We claim that the inverse $\tilde{H}$ has the following explicit form:

$$\tilde{H}_{(w,x),(y,z)} = -\frac{1}{L^2}(I + L \cdot M)_{x,y}(I + L \cdot M)_{w,z}$$

To show this, it's just a matter of verifying that $\sum_{y,z} H_{(w,x),(y,z)} \tilde{H}_{(y,z),(p,q)} = \delta((w, x), (p, q))$.

But this is easy enough:

$$\sum_{y,z} H_{(w,x),(y,z)} \tilde{H}_{(y,z),(p,q)} = \sum_{y,z} (-L^2(I + L \cdot M)^{-1}_{x,y}(I + L \cdot M)_{w,z})(-1/L^2(I + L \cdot M)_{y,q}(I + L \cdot M)_{z,p})$$
$$= \sum_{y} (I + L \cdot M)^{-1}_{x,y}(I + L \cdot M)_{y,q} \sum_{z} (I + L \cdot M)_{w,z}(I + L \cdot M)_{z,p}$$
$$= \delta(w, p)\delta(x, q) = \delta((w, x), (p, q))$$

This finishes the proof of Lemma 1.

□

3.2 Bounding γ

Finally, we want to estimate $\gamma = \max_{\vec{x} \in \mathcal{K}, \vec{\mathcal{P}}_t} \vec{\mathcal{P}}_t^\intercal [\nabla^2 \mathcal{R}(\vec{x})]^{-1} \vec{\mathcal{P}}_t$, which will be relatively easy. Given the form of $\vec{\mathcal{P}}_t$, we can write this as $\sum_{a,b,c,d} \mathcal{P}_{a,b} \mathcal{P}_{c,d} [\nabla^2 \mathcal{R}(\vec{x})]^{-1}_{(i_t,a),(j_t,b)}$ where (i_t, j_t) is the edge chosen at timestep t, and $\mathcal{P}_{a,b}$ is the payoff of playing label a on vertex i_t and label b on vertex j_t. So, we want to bound

$$\sum_{a,b,c,d} \mathcal{P}_{a,b} \mathcal{P}_{c,d} [\nabla^2 \mathcal{R}(\vec{x})]^{-1}_{((i_t,a),(j_t,b)),((i_t,c),(j_t,d))}$$

$$= \sum_{a,b,c,d} -\frac{1}{L^2} \mathcal{P}_{a,b} \mathcal{P}_{c,d} (I + L \cdot M)_{(j_t,b),(j_t,c)} (I + L \cdot M)_{(i_t,a),(i_t,d)}$$

However, since $\mathcal{P}_{a,b}, \mathcal{P}_{c,d} \in [-1, 1]$, it suffices to upper bound

$$\sum_{a,b,c,d} \frac{1}{L^2} (I + L \cdot M)_{(j_t,b),(j_t,c)} (I + L \cdot M)_{(i_t,a),(i_t,d)}$$

We can write

$$(I + L \cdot M)_{(j_t,b),(j_t,c)} (I + L \cdot M)_{(i_t,a),(i_t,d)} =$$

$$I_{(j_t,b),(j_t,c)} I_{(i_t,a),(i_t,d)} + L \cdot M_{(j_t,b),(j_t,c)} I_{(i_t,a),(i_t,d)} + I_{(j_t,b),(j_t,c)} L \cdot M_{(i_t,a),(i_t,d)} + L \cdot M_{(j_t,b),(j_t,c)} L \cdot M_{(i_t,a),(i_t,d)}$$

Let's examine each of the terms above, when we sum over all a, b, c, d.

In order for $I_{(j_t,b),(j_t,c)} I_{(i_t,a),(i_t,d)}$ to be non-vanishing, it must be the case that $b = c$, $a = d$. So

$$\sum_{a,b,c,d} I_{(j_t,b),(j_t,c)} I_{(i_t,a),(i_t,d)} = L^2$$

Similarly, for the second,

$$\sum_{a,b,c,d} L \cdot M_{(j_t,b),(j_t,c)} I_{(i_t,a),(i_t,d)} = \sum_{a,b,c} L \cdot M_{(j_t,b),(j_t,c)} = L^2 \sum_{b,c} M_{(j_t,b),(j_t,c)} = L^2$$

where the last equality is just due to the marginalization property of our M matrix. The third term is exactly the same as this one.

Finally for the last term,

$$\sum_{a,b,c,d} L \cdot M_{(j_t,b),(j_t,c)} L \cdot M_{(i_t,a),(i_t,d)} = L^2 \sum_{b,c} M_{(j_t,b),(j_t,c)} \sum_{a,d} M_{(i_t,a),(i_t,d)} = L^2$$

where the last equality is again due to marginalization.

Hence,

$$\gamma \leq \sum_{a,b,c,d} \frac{1}{L^2} (I + L \cdot M)_{(j_t,b),(j_t,c)} (I + L \cdot M)_{(i_t,a),(i_t,d)} \leq 4$$

which finishes the proof of Theorem 3.

□

4 Conclusion and open problems

In this paper we studied the online local learning framework. In particular, we studied the dependency on the label set size of the regret achievable in polynomial time. We showed that follow the regularized leader with a log-determinantal regularizer achieves regret $\sqrt{nLT}$, and we proved a matching lower bound - both based on planted clique and planted dense subgraph.

An interesting open problem is to investigate whether the regret bound can be improved when allowing sub-exponential time algorithms - since both planted clique and planted dense subgraph do admit sub-exponential time algorithms. A natural approach is to maintain higher order pseudo-moments, following similar approaches when using the Lasserre/Sum of Squares hierarchies.

A key difficulty here is the right choice of the regularizer. The log determinant regularizer is one particular approximation of the entropy of a distribution over the set of all possible labelings, matching the pseudo-moments that we maintain during the algorithm. Even if we one has access to higher order moments, it is not clear if there is a better candidate than the log determinant. If one were to take a page out of the graphical models literature, one would use generalizations of the Bethe entropy, commonly called Kikuchi approximations (Yedidia et al., 2003), which in this case do not work very well.

The log determinantal approximation used in (Christiano, 2014a) actually appeared even earlier in (Wainwright and Jordan, 2006) and roughly corresponds to the entropy of the Gaussian which matches a given matrix of second moments. It isn't clear how to extend this approach to higher moments, but any breakthroughs here might have wider ramifications by providing insights into designing new entropy approximations for doing approximate inference in graphical models.

Another open problem is basing the hardness of achieving regret $\sqrt{nLT}$ on more standard, worst case assumptions (e.g. NP-hardness, UGC-hardness). Indeed, it isn't obvious that randomness is *required* for proving hardness, but it does seem to help. This mirrors the current state of affairs in improper learning, where the only known hardness results are either based on cryptographic assumptions or very recently, refuting random DNF formulas (Daniely et al., 2014).

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A Full proof of Theorem 1

Theorem 1. *Let $\epsilon = \Omega(1)$. If regret $\sqrt{nL^{\beta}T}$ for*

$$\beta = \left(1 - \omega\left(\frac{1}{\log n}\right)\right)\left(\frac{1}{\frac{1}{2}+\epsilon}\right) - 1$$

is achievable in polynomial time, then one can distinguish between $G(n, 1/2)$ and $G\left(n, 1/2, n^{1/2-\epsilon}\right)$ with probability $\frac{4}{5}$ in polynomial time.

Proof. We produce an instance for the online local learning problem, given an instance of the planted clique problem with size of the planted clique k, in the following way.

We will randomly partition the graph we are given into $n' = n/l$ clusters, each containing l vertices, where $l = 10\frac{n}{k}$. We associate each vertex with a unique label in $\{1, .., l\}$. We then use this as an instance for the online learning problem as follows. We run it for $T = \binom{n'}{2}$ steps.

We query all pairs all pairs of clusters (C_1, C_2) in an arbitrary order. The algorithm responds with some labeling for the clusters (l_i, l_j), and the payoff is 1 if the vertex for l_i in C_1 has an edge to the vertex for l_j in C_2. Otherwise, the payoff is 0.

Let's assume the original graph was from $G(n, 1/2)$. Then, we claim that any algorithm (regardless if efficient or not) will get a payoff of at most $\frac{T}{2} + 5\frac{\sqrt{T}}{2}$ with probability at least $\frac{4}{5}$.

The above probability is with respect to the randomness in generating the graph from $G(n, 1/2)$, the partitioning of the vertices, and any randomness in the algorithm. If we denote the payoff as a random variable $\mathcal{P}$, the graph as a random variable $\mathcal{G}$, the partitioning as a random variable $\mathcal{R}_c$, and any randomness of the algorithm as $\mathcal{R}_a$, the joint distribution of $\mathcal{P}, \mathcal{G}, \mathcal{R}_c$ and $\mathcal{R}_a$ can be factored as

$$\mathbb{P}\left(\mathcal{P}, \mathcal{G}, \mathcal{R}_c, \mathcal{R}_a\right) = \mathbb{P}\left(\mathcal{G}\right)\mathbb{P}\left(\mathcal{R}_c|\mathcal{G}\right)\mathbb{P}\left(\mathcal{R}_a|\mathcal{G}, \mathcal{R}_c\right)\mathbb{P}\left(\mathcal{P}|\mathcal{R}_a, \mathcal{G}, \mathcal{R}_c\right)$$

But, since the partitioning is done independently of the graph, it follows that

$$\mathbb{P}\left(\mathcal{P}, \mathcal{G}, \mathcal{R}_c, \mathcal{R}_a\right) = \mathbb{P}\left(\mathcal{R}_p\right)\mathbb{P}\left(\mathcal{G}\right)\mathbb{P}\left(\mathcal{R}_a|\mathcal{G}, \mathcal{R}_c\right)\mathbb{P}\left(\mathcal{P}|\mathcal{R}_a, \mathcal{G}, \mathcal{R}_c\right)$$

By symmetry it's also clear that for any two partitionings $\mathcal{C}, \mathcal{C}'$,

$$\mathbb{P}\left(\mathcal{G}\right)\mathbb{P}\left(\mathcal{R}_a|\mathcal{G}, \mathcal{R}_c = \mathcal{C}\right)\mathbb{P}\left(\mathcal{P}|\mathcal{R}_a, \mathcal{G}, \mathcal{R}_c = \mathcal{C}\right) = \mathbb{P}\left(\mathcal{G}\right)\mathbb{P}\left(\mathcal{R}_a|\mathcal{G}, \mathcal{R}_c = \mathcal{C}'\right)\mathbb{P}\left(\mathcal{P}|\mathcal{R}_a, \mathcal{G}, \mathcal{R}_c = \mathcal{C}'\right)$$

Since we care about the payoff only, we can then fix some arbitrary partitioning $\mathcal{C}_0$ of the vertices, and argue about $\mathbb{P}\left(\mathcal{P}, \mathcal{G}, \mathcal{R}_a|\mathcal{R}_c = \mathcal{C}_0\right)$

To analyze this distribution, we perform one more piece of probabilistic trickstery with the principle of deferred decisions: namely we can assume the graph $\mathcal{G}$ is being generated as the online learning game progresses.

Let's make this more precise. Let's call the clusters produced by the partitioning $\mathcal{C}_0$: $\{C_1, C_2, \ldots, C_{n'}\}$. Then, we consider the following two random processes.

In the first process, we will assume that we generate a graph from $G(n, 1/2)$, and at time step t, for the pair of clusters (C_1^t, C_2^t) queried at that time step, we reveal the payoffs (equivalently edges) to the algorithm.In the second one, at each time step t, after the algorithm has made its decision, we randomly pick the edges between vertices in the clusters (C_1^t, C_2^t).

Let $\mathcal{X}_t$ be the random variable denoting the labels the algorithm chooses at the t-th step, let $\mathcal{E}_t$ be the random variable which denotes the edges between the vertices in (C_1^t, C_2^t), and let $\mathcal{G}_t$ be a random variable denoting the edges between all the pairs of clusters $(C_1^i, C_2^i), i \in [t]$.[4] Let's consider the joint distribution of these variables under $\mathbb{P}_1$, which will denote the first random process, and $\mathbb{P}_2$, which will denote the second.

We claim that

$$\mathbb{P}_1(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_t, \mathcal{G}_t) = \mathbb{P}_2(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_t, \mathcal{G}_t)$$

[4] Notice, since we are looking at one fixed partitioning $\mathcal{C}_0$, in both processes, all the random variables above are actually over the same domain.

For ease of notation, let's denote $\vec{\mathcal{X}_t} = (\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_t)$. We will show the claim by induction. Namely, for $i = \{1, 2\}$, by the chain rule:

$$\mathbb{P}_i(\vec{\mathcal{X}_t}, \mathcal{G}_t) = \mathbb{P}_i(\vec{\mathcal{X}_{t-1}}, \mathcal{G}_{t-1})\mathbb{P}_i(\mathcal{X}_t|\vec{\mathcal{X}_{t-1}}, \mathcal{G}_{t-1})\mathbb{P}_i(\mathcal{E}_t|\vec{\mathcal{X}_t}, \mathcal{G}_{t-1})$$

By induction, $\mathbb{P}_1(\vec{\mathcal{X}_{t-1}}, \mathcal{G}_{t-1}) = \mathbb{P}_2(\vec{\mathcal{X}_{t-1}}, \mathcal{G}_{t-1})$. However, $\mathbb{P}_i(\mathcal{X}_t|\vec{X_{t-1}}, \mathcal{G}_{t-1})$ only depends on the randomness of the algorithm, so it is equal for $i = \{1, 2\}$. Also, clearly, $\mathbb{P}_i(\mathcal{E}_t|\vec{\mathcal{X}_t}, \mathcal{G}_{t-1}) = \mathbb{P}_i(\mathcal{E}_t|\mathcal{G}_{t-1})$ since the graph is generated independently from the algorithm. But, since the graph is Erdös-Rényi, each of the edges is generated independently of any other edges. Hence, $\mathbb{P}_1(\mathcal{E}_t|\mathcal{G}_{t-1}) = \mathbb{P}_2(\mathcal{E}_t|\mathcal{G}_{t-1})$. So, we are done by induction.

In the second random process, if $\mathcal{P}_t$ is a random variable denoting the payoff at time t, clearly $\mathcal{P}_t$ is just a Bernoulli $0-1$ variable, which is 1 with probability $\frac{1}{2}$. Furthermore, the payoffs at different rounds are independent, so by Chernoff,

$$\Pr\left[\sum_{t=1}^{T} \mathcal{P}_t \geq \frac{T}{2} + 5\frac{\sqrt{T}}{2}\right] \leq e^{-50}$$

In particular, with probability $\geq \frac{4}{5}$, any algorithm gets payoff of $\frac{1}{2}\binom{k/10}{2} + o(k^2)$.

Let's proceed to the planted case. We claim that with probability at least $\frac{4}{5}$, there is a fixed labeling with payoff at least $\binom{2k/25}{2}$.

Let $\mathcal{I}_i$ be an indicator random variable for the event that no vertex from the planted clique belongs to cluster i.

The partitioning is done independently of the graph, so

$$\Pr[\mathcal{I}_i = 1] = \left(1 - \frac{10}{k}\right)^k \leq e^{-\frac{10}{k}k} = e^{-10}$$

Hence, if $\mathcal{I}$ is a random variable for the total number of clusters which contain no vertices from the planted clique, we know that

$$\mathbb{E}[\mathcal{I}] = \sum_{i=1}^{n'} \mathbb{E}[\mathcal{I}_i] \leq \frac{k}{10}e^{-10}$$

By Markov's inequality,

$$\Pr\left[\mathcal{I} \geq \frac{k}{50}\right] \leq \frac{\mathbb{E}[\mathcal{I}]}{\frac{k}{50}} \leq 5e^{-10} \leq \frac{1}{5}$$

So, with probability at least $\frac{4}{5}$, the number of clusters with at least one vertex from the planted clique is at least $\frac{k}{10} - \frac{k}{50} = \frac{2k}{25}$. In this case, the labeling where we label each of the clusters with a vertex from the planted clique gets a payoff of at least $\binom{2k/25}{2}$.

In this new online learning instance we constructed, the number of vertices is n', the number of rounds is T, and the label size is l. Let's suppose we can achieve regret of $\sqrt{n'l^{\beta}T}$. Whenever the original graph was a planted instance, the algorithm would get a payoff of at least

$$\binom{2k/25}{2} - \sqrt{n'l^{\beta}T}$$

with probability at least $\frac{4}{5}$, whereas in a random instance, we'd get a payoff at most

$$\frac{1}{2}\binom{k/10}{2} + o(k^2)$$

with probability at least $\frac{4}{5}$.

We claim that:

$$\binom{2k/25}{2} \geq \left(1 + \frac{1}{100}\right)\frac{1}{2}\binom{k/10}{2}$$

Indeed, $\binom{2k/25}{2} = \frac{2k/25(2k/25-1)}{2} \geq (1 - \frac{1}{100})\frac{(2k/25)^2}{2}$, for large enough k, and

$$\left(1 - \frac{1}{100}\right)\frac{(2k/25)^2}{2} \geq \left(1 + \frac{1}{100}\right)\frac{1}{2}\frac{(k/10)^2}{2} \geq \left(1 + \frac{1}{100}\right)\frac{1}{2}\binom{k/10}{2}$$

Hence, if $\sqrt{n' l^\beta T} = o(k^2)$, we can distinguish between a planted and random instance with probability $\frac{4}{5}$. We will show exactly that.

First we claim that

$$l^\beta = o\left(\frac{n}{l}\right) \tag{4}$$

Since $l = 10\frac{n}{k} = 10n^{\frac{1}{2}+\epsilon}$, after rearranging terms, 4 is equivalent to

$$n^{(\beta+1)\left(\frac{1}{2}+\epsilon\right)} = o(n)$$

Notice that $n^{\omega(\frac{1}{\log n})} = \omega(1)$, so for the above it is sufficient that

$$(\beta + 1)\left(\frac{1}{2} + \epsilon\right) = 1 - \omega\left(\frac{1}{\log n}\right)$$

But since

$$\beta = \left(1 - \omega\left(\frac{1}{\log n}\right)\right)\left(\frac{1}{\frac{1}{2} + \epsilon}\right) - 1$$

the above is clearly satisfied.

Hence,

$$\sqrt{n' l^\beta T} = \sqrt{\frac{n}{l} l^\beta \binom{\frac{n}{l}}{2}} = o\left(\sqrt{l^\beta \left(\frac{n}{l}\right)^3}\right) = o\left(\left(\frac{n}{l}\right)^2\right) = o(k^2)$$

which finishes the proof.

□

B Proofs for Section 2.2

Theorem 2. *Let ϵ, α, k satisfy the conditions of Conjecture 2. If regret $\sqrt{nL^\beta T}$ for*

$$\beta = 2\frac{\frac{1}{2} - \left(\frac{1}{2} - \epsilon'\right)(\alpha + \epsilon) - \omega\left(\frac{1}{\log n}\right)}{\frac{1}{2} + \epsilon'} - 1$$

is achievable in polynomial time, then one can distinguish between $G(n, p_s)$ and $G(n, p_s, k, p_d)$, where $p_s = n^{-\alpha}, k = n^{\frac{1}{2}-\epsilon'}, p_d = k^{-\alpha-\epsilon}$ with probability $\frac{4}{5}$ in polynomial time.

Proof. We proceed in the same way as in the proof of Theorem 1. Namely, we will partition our graph randomly into $n' = \frac{n}{l}$ clusters, each of size $\frac{n}{l}$, where $l = 10\frac{n}{k}$. As before, we will query all T pairs of clusters, and the payoff will be 1 if there is an edge between the labels supplied by the learner, and 0 otherwise.

As before, we claim that in the case when the graph is $G(n, p_s)$, with high probability, any algorithm can achieve at most

$$T \cdot p_s + 5\frac{\sqrt{T \cdot p_s}}{2} = \binom{k/10}{2} \cdot p_s + 5\frac{\sqrt{\binom{k/10}{2} \cdot p_s}}{2}$$

Again, by the principle of deferred decision, we can assume the edges between the clusters are revealed after the algorithm reveals its decision - so the payoff at any round is a Bernoulli variable with probability p_s. By Chernoff, if $\mathcal{P}_t$ is the payoff at round in this random process

$$\Pr\left[\sum_{t=1}^{T}\mathcal{P}_t \geq T\cdot p_s + 5\frac{\sqrt{T\cdot p_s}}{2}]\right] \leq 1/5$$

In the planted case, completely the same as in Theorem 1, with probability $1-5e^{-10}\geq\frac{9}{10}$, there will be at least $\frac{2k}{25}$ clusters which contain a vertex from the planted graph.

Conditioned on the above event happening, we claim that we'd get payoff of at least $\binom{2k/25}{2}\cdot p_d - 5\frac{\sqrt{\binom{2k/25}{2}\cdot p_d}}{2}$ with probability at least $\frac{8}{9}$. To show this, first notice that conditioned on belonging to two different clusters, the probability of an edge existing between two vertices in the planted graph is a Bernoulli $0-1$ variable, which is 1 with probability p_d. This is true since the partitioning is done independently from the graph. But then, the payoff is at least $\binom{2k/25}{2}\cdot p_d - 5\frac{\sqrt{\binom{2k/25}{2}\cdot p_d}}{2}$ with probability at least $1-e^{-50}\geq\frac{8}{9}$ by Chernoff.

Hence, in the planted case, again, with probability at least $\frac{4}{5}$, there is a fixed labeling with payoff at least $\binom{2k/25}{2}\cdot p_d - 5\frac{\sqrt{\binom{2k/25}{2}\cdot p_d}}{2}$.

But, since $p_s = o(p_d)$ if the regret is $\sqrt{n'l^\beta T}$, such that $\sqrt{n'l^\beta T} = o(k^2\cdot p_d)$, we can distinguish between the planted and random case with probability at least $\frac{4}{5}$.

Since $\frac{n}{l}=\Theta(k)$, it's sufficient to show:

$$\sqrt{n'l^\beta T} = o((\frac{n}{l})^2 k^{-\alpha-\epsilon}) \Leftrightarrow$$

$$l^{(\beta+1)/2} = o(n^{\frac{1}{2}-(\frac{1}{2}-\epsilon')(\alpha'+\epsilon)}) \tag{5}$$

Plugging in $l = 10\frac{n}{k} = 10n^{\frac{1}{2}+\epsilon'}$, 5 is equivalent

$$n^{\left(\frac{\beta+1}{2}\right)\left(\frac{1}{2}+\epsilon'\right)} = o(n^{\frac{1}{2}-(\frac{1}{2}-\epsilon')(\alpha'+\epsilon)})$$

As before, for this it's sufficient that,

$$\left(\frac{\beta+1}{2}\right)\left(\frac{1}{2}+\epsilon'\right) = \left(\frac{1}{2}-\left(\frac{1}{2}-\epsilon'\right)(\alpha+\epsilon)\right) - \omega\left(\frac{1}{\log n}\right)$$

It's easy to check for our choice of β that this is satisfied, which finishes the proof. ☐

Corollary 2. *Let $\epsilon',\alpha,\epsilon=\Omega(1)$ and $\alpha\geq\epsilon$. If we can achieve regret $\sqrt{nL^{1-\epsilon'-\alpha-\epsilon}T}$ in polynomial time, we can distinguish between $G(n,p_s)$ and $G(n,p_s,k,p_d)$ in polynomial time with probability $\frac{4}{5}$, where $p_s = n^{-\frac{\alpha}{8}}, k = n^{\frac{1}{2}-\frac{\epsilon'}{4}}, p_d = k^{-\frac{\alpha}{8}-\frac{\epsilon}{8}}$. In particular, if Conjecture 2 is true, no polynomial time algorithm can achieve regret $\sqrt{nL^{1-\delta}T}$, for any $\delta=\Omega(1)$.*

Proof. For notational ease, let $\tilde{\alpha}=\frac{\alpha}{8}$, $\tilde{\epsilon}=\frac{\epsilon}{8}$, $\tilde{\epsilon'}=\frac{\epsilon'}{4}$.

First, notice that $p_s=o(p_d)$. Indeed, since $p_s=n^{-\tilde{\alpha}}$ and $p_d=k^{-\tilde{\alpha}-\tilde{\epsilon}}$,

$$p_s = o(p_d) \Leftrightarrow n^{-\tilde{\alpha}} = o(n^{-(\frac{1}{2}-\tilde{\epsilon'})(\tilde{\alpha}+\tilde{\epsilon})})$$

However, since $\alpha\geq\epsilon$,

$$\tilde{\alpha} \geq \frac{1}{2}(\tilde{\alpha}+\tilde{\epsilon}) = (\frac{1}{2}-\tilde{\epsilon'})(\tilde{\alpha}+\tilde{\epsilon}) + \tilde{\epsilon'}(\tilde{\alpha}+\tilde{\epsilon})$$

Since $\tilde{\epsilon'},\tilde{\alpha},\tilde{\epsilon}=\Omega(1)$, clearly this implies $n^{-\tilde{\alpha}} = o(n^{-(\frac{1}{2}-\tilde{\epsilon'})(\tilde{\alpha}+\tilde{\epsilon})})$

Since clearly $\tilde{\epsilon},\tilde{\epsilon'},\tilde{\alpha}=\omega(\frac{1}{\log n})$, directly applying Theorem 2, to distinguish between $G(n,p_s)$ and $G(n,p_s,k,p_d)$, where $k=n^{\frac{1}{2}-\tilde{\epsilon'}}, p_s=n^{-\tilde{\alpha}}$ and $p_d=k^{-\tilde{\alpha}-\tilde{\epsilon}}$, achieving regret $\sqrt{nL^\beta T}$ is sufficient, for

$$\beta = 2\frac{\frac{1}{2}-2(\frac{1}{2}+\tilde{\epsilon'})(\tilde{\alpha}+\tilde{\epsilon})}{\frac{1}{2}+\tilde{\epsilon'}} - 1 = \frac{1-4(\frac{1}{2}+\tilde{\epsilon'})(\tilde{\alpha}+\tilde{\epsilon})}{\frac{1}{2}+\tilde{\epsilon'}} - 1$$

$$= 1 - \frac{2\tilde{\epsilon'} + 4(\frac{1}{2} + \tilde{\epsilon'})(\tilde{\alpha} + \tilde{\epsilon})}{\frac{1}{2} + \tilde{\epsilon'}} \geq 1 - 4\tilde{\epsilon'} - 8(\frac{1}{2} + \tilde{\epsilon'})(\tilde{\alpha} + \tilde{\epsilon}) \geq 1 - 4\tilde{\epsilon'} - 8\tilde{\alpha} - 8\tilde{\epsilon}$$

where the next to last inequality holds since $\tilde{\epsilon'} \geq 0$ and the last since $\tilde{\epsilon'} \leq \frac{1}{2}$.

So, if we can achieve regret

$$\sqrt{nL^{1-4\tilde{\epsilon'}-8\tilde{\alpha}-8\tilde{\epsilon}}T} = \sqrt{nL^{1-\epsilon'-\alpha-\epsilon}}$$

we can distinguish between $G(n, p_s)$ and $G(n, p_s, k, p_d)$, as we needed.

□