CS 536: Homework 0

1 Problem 1

Let X be a discrete random variable with support in non negative integers. Show that $E[X] = \sum_{x=0}^{\infty} Pr[X \geq x]$.

2 Problem 2

Let X be a real random variable with uniform distribution on $(-1, 2)$ and let $Y = X^2$. Find the density of Y.

3 Problem 3

Let $X \in \mathbb{R}^d$. We say that X is normally distributed, i.e., $X \sim N(\mu, \Sigma)$ if,

$$p(x) = \frac{1}{(2\pi)^{d/2}|\Sigma|} \exp(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu))$$

Here μ is $d \times 1$ and Σ is $d \times d$. If X is normally distributed then show that

- $E[X] = \mu, Cov(X) = \Sigma$.
- If c is a scalar then, $cX \sim N(c\mu, c^2\Sigma)$.
- If A is $p \times d$ and b is $p \times 1$, then $AX + b \sim N(A\mu + b, A\Sigma A^T)$.

4 Problem 4

Suppose that $X \sim N(\mu, \Sigma)$ is a d dimensional random vector. Let $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma 12 \\ \Sigma 21 & \Sigma 22 \end{pmatrix}$. Here X_1 is $p \times 1$, μ_1 is $p \times 1$ and Σ_{11} is $p \times p$. Show that

- $X_1 \sim N(\mu_1, \Sigma_{11}), X_2 \sim N(\mu_2, \Sigma_{22})$.
- X_1 and X_2 are independent if and only if $\Sigma_{12} = 0$.
- If $\Sigma_{22} > 0$, then the distribution of X_1 conditioned on X_2 is
$$X_1|X_2 \sim N(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(X_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$$

5 Problem 5

Let $X_1, X_2, \cdots X_n$ be independent and identically distributed (i.i.d.), one dimensional Normal random variables with $E[x_i] = \mu$, and $Var(X_i) = \sigma^2$, i.e. $X_i \sim N(\mu, \sigma^2)$. Define the sample mean as $\bar{X}_n = \frac{1}{n}\sum_i X_i$ and the sample variance as $S_n^2 = \frac{1}{n-1}\sum_i (x_i - \bar{X}_n)^2$. Show that

- $\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$.

- $\frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1}$, a chi-squared distribution with $n-1$ degrees of freedom.
- $\bar{X}_n$ and S_n^2 are independent.