CS 205 Sections 07 and 08.
Midterm 2 – 4/14/04.
5 of 6 questions, 2 pages, 80 minutes, 100 points.

Answer questions 1 and 2 and **any three** of the four questions 3–6 in a blue book. Start each question on a new page. Label the questions clearly and prominently. Don't forget to fill in your name and the class on the cover page of each blue book you use!

1. (12 points) Let A, B, C, D, and E be nonempty sets. Suppose we have three functions $f : A \rightarrow B$, $g : C \rightarrow D$ and $h : B \times D \rightarrow E$.

   Using whatever notation you like, define a new function $m : A \times C \rightarrow E$ using f, g, and h.

   Answer:

   $$m(x,y) = h(f(x), g(y)).$$

2. (22 points) Let A, B, and C be nonempty sets such that $B \subset C$. Suppose we have a one-to-one and onto function $f : A \to B$. Consider the function $g : A \to C$ defined by $g(x) = f(x)$.

   (a) Is g one-to-one? Prove your answer.

   Answer:

   Yes, g is one-to-one.
   To prove this, we need to show

   $$\forall x \in A \forall y \in A (g(x) = g(y) \to x = y)$$

   The argument is as follows. Let $a \in A$ and $b \in A$ be arbitrary elements of A, and suppose $g(a) = g(b)$. By the definition of g, $g(a) = f(a)$ and $g(b) = f(b)$, so $f(a) = f(b)$. But since f is one-to-one, $a = b$. This shows g is also one-to-one.

   (b) Is g onto? Prove your answer.

   Answer:

   No, g is not onto.
   To show this, we need to show

   $$\exists y \in C \neg \exists x \in A (g(x) = y)$$

   Since $B \subset C$, there is some $c \in C \setminus B$. Suppose for indirect proof that for some $x \in A$, $g(x) = c$. For the sake of argument, call it a. By the definition of g, $c = g(a) = f(a)$. Since $f : A \to B$, $c \in B$. But we assumed $c \notin B$. This is a contradiction: So $\neg \exists x \in A (g(x) = c)$. So g is not onto.

3. (22 points if selected) Recall that when a and c are integers, we write $a|c$ when $ax = c$ for some integer x. In class we showed that when a and b are positive integers such that $\gcd(a,b) = 1$ and c is an integer, if $a|bc$ then $a|c$. Formally:

$$\forall a \forall b \forall c (a > 0 \wedge b > 0 \wedge \gcd(a,b) = 1 \wedge a|bc \rightarrow a|c)$$

Use this fact to show that if a and b are positive integers such that $\gcd(a,b) = 1$ and d is an integer, then if $a|d$ and $b|d$ then $ab|d$. In other words, assume

$$a > 0, b > 0, \gcd(a,b) = 1, a|d \text{ and } b|d$$

(where all the variables are integers). You should argue mathematically and logically from these assumptions to the conclusion that

$$ab|d$$

Your answer should be a precise mathematical argument mixing English and mathematical notation as you see fit.

Hint: Use the definition of $b|d$ to figure out what you know about d.

Answer:

The proof is as follows:

1	$b\|d$	Assumed
2	$bx = d$	Definition of |
3	$a\|bx$	We assume $a\|d$
4	$a\|x$	By the key fact stated in the problem
5	$ay = x$	Definition of |
6	$d = aby$	Subtitution into 2, and algebra
7	$ab\|d$	Definition of |

4. (22 points if selected) True or false: whenever m and n are positive integers and a, b, c and d are integers, if $a \equiv b \pmod m$ and $c \equiv d \pmod n$ then $ac \equiv bd \pmod{mn}$. Justify your answer with a precise mathematical argument.

   Hint: See whether the statement is true or false on several examples before you decide what your answer will be.

   Answer:

   False.

   There are many counterexamples. For example:

$$
\begin{gathered}
1 \equiv 3 (\mathrm{mod} 2) \\
1 \equiv 5 (\mathrm{mod} 4) \\
1 \not\equiv 15 (\mathrm{mod} 8)
\end{gathered}
$$

5. (22 points if selected) Prove the following statement by mathematical induction for all integers $n \geq 0$:

$$\sum_{k=0}^{n} \left(-\frac{1}{2}\right)^k = \frac{2}{3} + \frac{1}{3} \cdot \left(-\frac{1}{2}\right)^n$$

Hint: The inductive property $P(n)$ that you are proving has the form $S(n) = T(n)$, where

$$S(n) = \sum_{k=0}^{n} \left(-\frac{1}{2}\right)^k$$

and

$$T(n) = \frac{2}{3} + \frac{1}{3} \cdot \left(-\frac{1}{2}\right)^n$$

Feel free to use these abbreviations to write your proof more compactly. Also, remember that

$$\left(-\frac{1}{2}\right)^t = -2 \cdot -\frac{1}{2} \cdot \left(-\frac{1}{2}\right)^t = -2\left(-\frac{1}{2}\right)^{t+1}$$

Answer:

First we show $P(0)$.

$$S(0) = \left(-\tfrac{1}{2}\right)^0 = 1$$
$$T(0) = \tfrac{2}{3} + \tfrac{1}{3} \cdot \left(-\tfrac{1}{2}\right)^0 = \tfrac{2}{3} + \tfrac{1}{3} = 1$$

Now we assume $P(n)$. We prove $P(n+1)$.

$$\begin{aligned}
S(n+1) \quad &= \textstyle\sum_{k=0}^{n+1} \left(-\tfrac{1}{2}\right)^k \\
&= \left(-\tfrac{1}{2}\right)^{n+1} + \textstyle\sum_{k=0}^{n} \left(-\tfrac{1}{2}\right)^k \\
&= \left(-\tfrac{1}{2}\right)^{n+1} + \tfrac{2}{3} + \tfrac{1}{3} \cdot \left(-\tfrac{1}{2}\right)^n \quad \text{(induction hypothesis)} \\
&= \tfrac{2}{3} + \left(-\tfrac{1}{2}\right)^{n+1} - \tfrac{2}{3} \cdot \left(-\tfrac{1}{2}\right)^{n+1} \quad \text{(algebra)} \\
&= \tfrac{2}{3} + \tfrac{1}{3}\left(-\tfrac{1}{2}\right)^{n+1} \\
&= T(n+1)
\end{aligned}$$

By induction, we conclude $P(n)$ for all $n \geq 0$.

6. (22 points if selected) Consider the following program:

$i := 0$
while $i \leq x - 1$
    $i := i + 1$

Suppose that the type of x is a real number and the type of i is an integer. This gives the domain of discourse for these variables. Initially, assume that $x \geq 0$. Show if this initial assertion is true, then when this program terminates, $i = \lfloor x \rfloor$.

Hint: Use the loop invariant that $i \leq x$. In addition, recall from class that

$$i = \lfloor x \rfloor \leftrightarrow i \leq x < i + 1$$

You can use this fact directly here.

Answer:

There are three things to show.

First, we show that before the loop is run, $i \leq x$. That's true because we assume that initially $x \geq 0$ and then the assignment gives $i = 0$.

Second, we show that we have a correct loop invariant. We assume that the loop invariant is true initially and the loop runs: $i \leq x \wedge i \leq x - 1$. Since initially $i \leq x - 1$, after we set $i := i + 1$, we have $i - 1 \leq x - 1$ or $i \leq x$. So the loop invariant is indeed true after the loop runs.

Third, we show that when the loop terminates, $i = \lfloor x \rfloor$. After the loop terminates, we have $i \leq x$ by the loop invariant. We also know $i \not\leq x - 1$, because the loop test must have failed. That means $i > x - 1$ or $i + 1 > x$. But $i \leq x < i + 1$ is equivalent to $i = \lfloor x \rfloor$.

Together, this shows that if the initial assertion is true, then when the program terminates, $i = \lfloor x \rfloor$.