CS 205 Sections 07 and 08.
Midterm 1 – 3/3/04.
8 questions, 5 pages, 150 points. Your name: ________________________________

1. (24 points) Fill in the truth table below:

P	Q	R	$P \leftrightarrow Q$	$\neg Q \vee R$	$(P \leftrightarrow Q) \rightarrow (\neg Q \vee R)$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

Answer:

P	Q	R	$P \leftrightarrow Q$	$\neg Q \vee R$	$(P \leftrightarrow Q) \rightarrow (\neg Q \vee R)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	T

2. (18 points) Show these formulas are logically equivalent using laws for logical equivalence.

(a) $\neg(P \wedge Q \wedge \neg R)$ $\quad$ $P \wedge Q \rightarrow R$

Answer:

$\neg(P \wedge Q \wedge \neg R)$	
$\neg(P \wedge Q) \vee \neg\neg R$	De Morgan's law
$\neg(P \wedge Q) \vee R$	Double negation
$(P \wedge Q) \rightarrow R$	Implication

(b) $\neg\forall x(P(x) \rightarrow \neg Q(x))$ $\quad$ $\exists x(P(x) \wedge Q(x))$

Answer:

$\neg\forall x(P(x) \rightarrow \neg Q(x))$	
$\exists x\neg(P(x) \rightarrow \neg Q(x))$	Quantifier and negation
$\exists x\neg(\neg P(x) \vee \neg Q(x))$	Implication
$\exists x(\neg\neg P(x) \wedge \neg\neg Q(x))$	De Morgan
$\exists x(P(x) \wedge Q(x))$	Double negation

3. (24 points) This question uses the predicates S, P and J:

   $S(x)$ represents the proposition that x is a sandwich.
   $P(x)$ represents the proposition that x has peanut butter.
   $J(x)$ represents the proposition that x has jelly.

   Represent the following statements in logic:

   (a) Everything with jelly or peanut butter is a sandwich.
   Answer:
   $$\forall x(J(x) \vee P(x) \to S(x))$$

   (b) No sandwich has peanut butter without jelly.
   Answer:
   $$\neg\exists x(S(x) \wedge P(x) \wedge \neg J(x))$$

   (c) Some sandwich has jelly without peanut butter.
   Answer:
   $$\exists x(S(x) \wedge J(x) \wedge \neg P(x))$$

4. (18 points) Now let SS be the set of sandwiches, PB be the set of things with peanut butter and JJ be the set of things with jelly. Give a mathematical expression that states the relationships among these sets if you know:

   (a) Everything with jelly or peanut butter is a sandwich.
   Answer:
   $$JJ \cup PB \subseteq SS$$

   (b) No sandwich has peanut butter without jelly.
   Answer:
   $$SS \cap (PB - JJ) = \emptyset$$

   (c) Some sandwich has jelly without peanut butter.
   Answer:
   $$SS \cap (JJ - PB) \neq \emptyset$$

5. (16 points) True or false:

$$\forall S \forall a (\{a\} \in P(S) \leftrightarrow a \in S)$$

In other words, $\{a\}$ is in $P(S)$, the power set of S, exactly when $a \in S$. Justify your answer with a *short*, *careful* mathematical argument using English and formal notation as you like.

Answer:

True.

By the definition of the power set, $\{a\} \in P(S)$ implies $\{a\} \subseteq S$. By the definition of subsets, $\{a\} \subseteq S$ implies $a \in S$. Conversely, $a \in S$ implies $\{a\} \subseteq S$, and $\{a\} \subseteq S$ implies $\{a\} in P(S)$.

6. (16 points) True or false:

$$\neg \exists S \exists x (x \in S \land x \subseteq S)$$

In other words, you can't have a set x be both a member and a subset of the same set S. Justify your answer with a *short*, *careful* mathematical argument mixing English and formal notation as you like.

Answer:

False.

There is an S, namely $\{\emptyset\}$ and an x, namely $\emptyset$ such that $x \in S \land x \subseteq S$. By the definition of element, $\emptyset \in \{\emptyset\}$. By the definition of $\emptyset$, $\emptyset$ has no elements. Thus all the elements of $\emptyset$ are elements of $\{\emptyset\}$. So $\emptyset \subseteq \{\emptyset\}$.

7. (16 points) Make the two assumptions:

$$\forall x Q(x)$$
$$\exists x P(x)$$

Give a formal proof of

$$\exists x(P(x) \wedge Q(x))$$

Answer:

1	$\forall x Q(x)$	Premise
2	$\exists x P(x)$	Premise
3	$\quad P(a)$	Premise for existential instantiation 2
4	$\quad Q(a)$	Universal instantiation 1
5	$\quad P(a) \wedge Q(a)$	Conjunction 3,4
6	$\quad \exists x(P(x) \wedge Q(x))$	Existential generalization 5
7	$\exists x(P(x) \wedge Q(x))$	Existential instantiation 2,3–6

8. (18 points) Make the assumption:

$$\forall x \forall y (S(x) \wedge A(x,y) \rightarrow \neg A(y,x))$$

(A shark only attacks something that won't attack it.)

Give a formal proof of

$$\forall x (A(x,x) \rightarrow \neg S(x))$$

(Anything that attacks itself isn't a shark.)

Answer:

1	$\forall x \forall y (S(x) \wedge A(x,y) \rightarrow \neg A(y,x))$	Premise
2	$\quad A(s,s)$	Premise for conditional proof
3	$\quad \forall y (S(s) \wedge A(s,y) \rightarrow \neg A(y,s))$	Universal Instantiation, 1
4	$\quad S(s) \wedge A(s,s) \rightarrow \neg A(s,s)$	Universal Instantiation, 3
5	$\qquad S(s)$	Premise for indirect proof
6	$\qquad S(s) \wedge A(s,s)$	Conjunction 2, 5
7	$\qquad \neg A(s,s)$	Modus ponens 4,6
8	$\qquad$ FALSE	Contradiction 2,7
9	$\quad \neg S(s)$	Indirect proof, 4–8
10	$A(s,s) \rightarrow \neg S(s)$	Conditional proof 2–9
11	$\forall x (A(x,x) \rightarrow \neg S(x))$	Universal generalization 10