CS 205 Sections 07 and 08.
Final – 5/10/04.
8 questions – 100 points – 180 minutes.

Answer all eight questions in a blue book. Start each question on a new page. Label the questions clearly and prominently. Don't forget to fill in your name and the class on the cover page of each blue book you use!

1. (10 points) Find truth-values for the propositional variables p, q and r so that

$$(p \leftrightarrow q) \rightarrow r$$

has a different truth-value from

$$\neg p \vee \neg q \vee r$$

Answer:

p is false, q is false, r is false.
In this case, $(p \leftrightarrow q) \rightarrow r$ is false but $\neg p \vee \neg q \vee r$ is true.

2. (15 points) Make the assumption:

$$\forall x \exists y Q(x,y)$$

and the assumption

$$\forall x \forall y \forall z (Q(x,y) \wedge Q(y,z) \rightarrow Q(y,y))$$

Using these assumptions, give a formal proof in predicate logic for the conclusion

$$\exists y Q(y,y)$$

Answer:

1	$\forall x \exists y Q(x,y)$	Premise
2	$\forall x \forall y \forall z (Q(x,y) \wedge Q(y,z) \rightarrow Q(y,y))$	Premise
3	$\exists y Q(a,y)$	UI, 1
4	$\quad Q(a,b)$	Premise for EI, 3
5	$\quad \exists y Q(b,y)$	UI, 1
6	$\quad\quad Q(b,c)$	Premise for EI, 5
7	$\quad\quad Q(a,b) \wedge Q(b,c)$	Conjunction, 4, 6
8	$\quad\quad Q(a,b) \wedge Q(b,c) \rightarrow Q(b,b)$	UI, 2 (three times)
9	$\quad\quad Q(b,b)$	Modus Ponens 7, 8
10	$\quad\quad \exists y Q(y,y)$	EG, 9
11	$\quad \exists y Q(y,y)$	EI, 5, 10
12	$\exists y Q(y,y)$	EI, 3, 11

3. (10 points) Assume that $A \subseteq B$ and $C \subseteq D$. Prove that $A \times C \subseteq B \times D$. Present your reasoning in a precise, detailed mathematical argument that considers an arbitrary element of $A \times C$.

   Answer:

   > $A \times C = \{(a,c) : a \in A \wedge c \in C\}$. Consider $(a,c) \in A \times C$. Then $a \in A$. Since $A \subseteq B$, $a \in B$. Meanwhile, $c \in C$, and since $C \subseteq D$, $c \in D$. So by the definition of Cartesian product, $(a,c) \in B \times D$. So by the definition of $\subseteq$, $A \times C \subseteq B \times D$.

4. (15 points) Consider a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined as follows:

$$f(x,y) = (x+y)^2 + x$$

   Prove that f is one-to-one, in four parts.

   (a) Start by stating what you must assume about two pairs of natural numbers (x,y) and (x',y') and what you must prove about them to show that f is one-to-one.

   Answer:

   > You must assume $f(x,y) = f(x',y')$. You must prove $(x,y) = (x',y')$.

   (b) We argue by cases, based on whether (1) $x+y = x'+y'$, (2) $x+y < x'+y'$, or (3) $x+y > x'+y'$. First give a direct proof in case $x+y = x'+y'$.

   Answer:

   > Assume $x+y = x'+y'$. We know $f(x,y) = (x+y)^2 + x = f(x',y') = (x'+y')^2 + x'$. Since $x+y = x'+y'$, it follows that $x = x'$, and then that $y = y'$.

   (c) Now, as background, show that

$$(x+y)^2 \leq f(x,y) < ((x+y)+1)^2$$

   Answer:

   > $f(x,y) = (x+y)^2 + x$.
   > Since $x \geq 0$, $f(x,y) \geq (x+y)^2$.
   > $((x+y)+1)^2 = (x+y)^2 + 2(x+y) + 1$. Since $x \geq 0$ and $y \geq 0$, $x < 2(x+y)+1$ so $f(x,y) < ((x+y)+1)^2$.

   (d) Next, give a proof in case $x+y < x'+y'$ using the background observation from (c). (This proof can either be a direct proof or a proof by contradiction.)

   Answer:

   > Assume $x+y < x'+y'$. Then $x+y+1 \leq x'+y'$. So $f(x,y) < (x'+y')^2$ by (c). But $(x'+y')^2 \leq f(x',y')$ by (c). So $f(x,y) < f(x',y')$. This contradicts the assumption that $f(x,y) = f(x',y')$.

   (This is really all you need, because the case for $x+y > x'+y'$ is exactly the same as what you have in (d), but with the places of (x,y) and (x',y') swapped. Not an exam question: Are you surprised to find a one-to-one function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$? I don't know. But the first time I saw it, I was.)

5. (10 points) Consider the function $d : \mathbb{N} \to \mathbb{N}$ where $d(n)$ is the sum of the digits in the decimal expansion of n. For example, $d(217) = 2+1+7 = 10$. Use mathematical induction over the natural numbers to prove $d(n) \equiv n (\text{mod } 3)$ for all $n \geq 0$.

   Hint. When you add one to n in decimal notation, you add one to the last digit in n that's not a 9, and change any subsequent 9s to 0s.

   Answer:

   For the base case, $n = 0$ and $d(n) = 0$. So we immeidately get $0 \equiv 0 (\text{mod } 3)$.

   For the inductive case, assume $d(n) \equiv n (\text{mod } 3)$ and consider $n+1$. Let k be the number of 9s that n ends with. Then $d(n+1) = d(n) + 1 - 9k$. We have $d(n) \equiv n (\text{mod } 3)$ by hypothesis. That means $d(n) = n + 3m$. So $d(n) + 1 = n+1+3m$. And $d(n+1) = d(n)+1-9k = n+1+3m-9k = n+1+3(m-3k)$. So $d(n+1) \equiv n+1 (\text{mod } 3)$.

   Since we know $d(n) \equiv n (\text{mod } 3)$ holds for $n = 0$, and we know that if it holds for n it holds for $n+1$, we conclude by induction that it holds for all $n \geq 0$.

6. (15 points) Consider the following program:

   $s := 0$
   $i := 0$
   while $i \leq n$ **do**
       $i := i+1$
       $s := s+m$
   end

   Suppose that n, m, s and i are all natural numbers. Is the following formula a correct invariant for the **while** loop:

   $$s = mi \wedge i \leq n$$

   Justify your answer with a mathematical argument. If it is a loop invariant, show that it is a loop invariant. If it is not, give a counterexample.

   Answer:

   This is not a correct loop invariant. When $i = n$, the loop runs. Afterwards, $i = n+1$. So it is no longer the case that $i \leq n$, as the purported invariant requires.

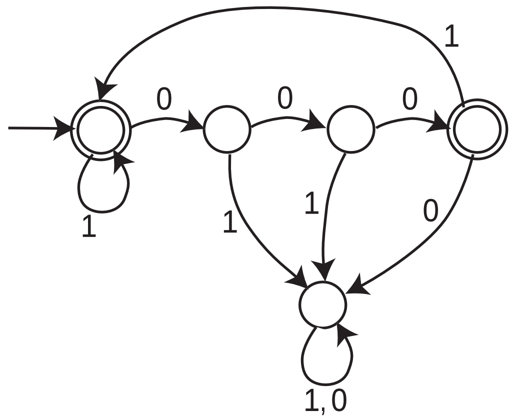
7. (15 points) The next two problems assume an input vocabulary $V = \{0, 1\}$. They consider a language L defined as follows:

   $$L = \{w \in V* \mid \text{each occurrence of 0 in } w \text{ is part of a sequence of exactly 3 0's in a row } \}$$

   For example, strings in L include λ, 111, 000, 110001000, and 11100011000111. Strings not in L include 101, 0010011 and 000000.

Draw a *deterministic* finite automaton that accepts the language L. Include an explicit "rejection" state, so that each state has exactly one outgoing edge for each possible input symbol. Remember to mark the start state and the final states.

Answer:



8. (10 points) Give a regular expression that corresponds to the language L.

   Hint. First write a regular expression for strings that begin and end with blocks of 000, where any successive blocks are separated by at least one 1. Then use this to write a regular expression for strings that may contain 1s at the beginning and end, and may contain 000s in blocks in the middle.

   Answer:

   $$1^*(\lambda|000(1^+000)^*)1^*$$