CS 205 Sections 07 and 08
Homework 3 Answers

1. Each of the following items gives a condition on a function. Construct a function satisfying that condition. The domain and codomain of your function must be chosen from the sets

$$U = \{a,b,c\}, V = \{x,y,z\}, W = \{1,2\}$$

   (a) One-to-one but not onto.
   Answer:
   $f : W \to U$ where $f(1) = a$ and $f(2) = b$.

   (b) Onto but not one-to-one.
   Answer:
   $f : U \to W$ where $f(a) = 1$, $f(b) = 1$ and $f(c) = 2$.

   (c) One-to-one and onto.
   Answer:
   $f : U \to V$ where $f(a) = x$, $f(b) = y$ and $f(c) = z$.

   (d) Neither one-to-one nor onto.
   Answer:
   $f : U \to V$ where $f(a) = x$, $f(b) = x$ and $f(c) = x$.

2. Each of the following items specifies a function $f : \mathbb{N} \to \mathbb{N}$, and specifies certain of its properties. In each case, give a precise mathematical argument showing that the function satisfies the properties.

   (a) $f(x) = 2x$ — one-to-one but not onto.
   Answer:
   To show f is one-to-one, we suppose $f(x) = f(y)$. By the definition of f, this means $2x = 2y$. Dividing both sides by 2, by algebra, we conclude $x = y$. Thus f is one-to-one.
   To show f is not onto, we note that $1 \in \mathbb{N}$ but $1 \neq 2x$ for $x \in \mathbb{N}$.

   (b) $f(x) = \lfloor x/2 \rfloor$ — onto but not one-to-one.
   Answer:
   To show f is onto, we consider $y \in \mathbb{N}$. Then $2y \in \mathbb{N}$. Moreover, $f(2y) = \lfloor 2y/2 \rfloor = \lfloor y \rfloor = y$. So there is an $x \in \mathbb{N}$ such that $f(x) = y$. So f is onto.
   To show f is not one-to-one, we note that $0 \in \mathbb{N}$ and $1 \in \mathbb{N}$ but $f(0) = \lfloor 0/2 \rfloor = 0$ and $f(1) = \lfloor 1/2 \rfloor = 0$.

   (c) $f(x) = \begin{cases} x-1 & \text{if } x \text{ is odd} \\ x+1 & \text{otherwise} \end{cases}$ — one-to-one and onto.
   Answer:
   First we make two observations. If x is odd then $f(x) = x - 1$ and $f(x)$ is even. If x is even then $f(x) = x + 1$ and $f(x)$ is odd.

To show that f is one-to-one, suppose $f(x) = f(y)$. Either $f(x)$ is odd or $f(x)$ is even. If $f(x)$ is odd, then by our observations, $f(x) = x+1$, and $f(y) = y+1$. So $x+1 = y+1$ and by algebra $x = y$. Likewise, if $f(x)$ is even, then by our observations $f(x) = x-1$ and $f(y) = y-1$. So $x-1 = y-1$ and by algebra $x = y$. This shows f is one-to-one.

To show that f is onto, consider $y \in \mathbb{N}$. If y is even, then $y+1 \in \mathbb{N}$ and $y = f(y+1)$. (Since $y+1$ is odd, $f(y+1) = (y+1)-1 = y$.) If y is odd, then $y-1 \in \mathbb{N}$ and $y = f(y-1)$. (Since $y-1$ is even, $f(y-1) = (y-1)+1 = y$.)

3. Let A, B and C be nonempty sets, and let $g : A \to B$ and $h : A \to C$ and let $f : A \to B \times C$ be defined by

$$f(x) = (g(x), h(x))$$

Give a precise mathematical argument for each of the following statements.

(a) If f is onto, then g and h are onto.

Answer:
Suppose f is onto. Then for all (b,c) in $B \times C$, there is an $a \in A$ such that $f(a) = (b,c)$. Let $u \in B$. Since C is nonempty there is some $v \in C$ and $(u,v) \in B \times C$. There is some $y \in A$ such that $f(y) = (u,v)$. This means that $g(y) = u$. So g is onto.

Likewise, for $v \in C$, since B is nonempty there is some $u \in B$ and $(u,v) \in B \times C$. So there is some $y \in A$ such that $f(y) = (u,v)$. This means $h(y) = v$ so h is onto.

(b) It is not the case that f must be onto whenever g and h are onto.

Answer:
We give a counterexample. We take $A = B = C = \mathbb{N}$, and let $g(x) = x$ and $h(x) = x$. Clearly g and h are onto. But f is not onto, because if $u \neq v$ there is no $y \in \mathbb{N}$ such that $f(y) = (u,v)$.

(c) If either g is one-to-one or h is one-to-one, then f is one-to-one.

Answer:
Suppose $f(x) = f(y)$. This means $(g(x), h(x)) = (g(y), h(y))$, and therefore $g(x) = g(y)$ and $h(x) = h(y)$. If g is one-to-one and $g(x) = g(y)$ then $x = y$. If h is one-to-one and $h(x) = h(y)$ then $x = y$. Thus, since either g is one-to-one or h is one-to-one, it follows that $x = y$. So f is one-to-one.

(d) It is possible for f to be one-to-one without either g or h being one-to-one.

Answer:
We give a counterexample. We take $A = \{0,1,2,3\}, B = \{0,1\}, C = \{0,1\}$. We consider the function g such that $g(0) = g(1) = 0$ and $g(2) = g(3) = 1$. We consider the function h such that $h(0) = h(2) = 0$ and $h(1) = h(3) = 1$. Clearly g and h are not one-to-one. However, in this case f is as follows: $f(0) = (0,0)$, $f(1) = (0,1)$, $f(2) = (1,0)$ and $f(3) = (1,1)$ so f is one-to-one.

4. Prove or disprove each of these statements about the floor and ceiling functions.

(a) For all real numbers x,

$$\lfloor \lceil x \rceil \rfloor = \lfloor x \rfloor$$

Answer:
We disprove this by giving a counterexample: 1.5. $\lceil 1.5 \rceil = 2$. So $\lfloor \lceil x \rceil \rfloor = 2$. But $\lfloor 1.5 \rfloor = 1$.

(b) $\lfloor x \rfloor = \lceil x \rceil$ if and only if x is an integer.

Answer:
Proof. If x is an integer, then the greatest integer no greater than x is x. So $\lfloor x \rfloor = x$. Meanwhile, the least integer no smaller than x is x so $\lceil x \rceil = x$. So $\lfloor x \rfloor = \lceil x \rceil$.

Conversely, suppose x is not an integer. Then $\lfloor x \rfloor < x$ while $x < \lceil x \rceil$. So $\lfloor x \rfloor < \lceil x \rceil$ and thus $\lfloor x \rfloor \neq \lceil x \rceil$.

(c) For all positive integers r,

$$\left\lfloor \log_2 \left\lfloor \frac{r+1}{2} \right\rfloor \right\rfloor = \left\lfloor \log_2 \left(\frac{r+1}{2} \right) \right\rfloor$$

Answer:
Consider a positive integer r. We will set n to be the lowest integer such that $r < 2^n$. We consider two cases separately. First, $r = 2^n - 1$. Then $\frac{r+1}{2} = 2^{n-1}$. This is an integer, so $\lfloor \frac{r+1}{2} \rfloor = 2^{n-1}$. This guarantees that

$$\left\lfloor \log_2 \left\lfloor \frac{r+1}{2} \right\rfloor \right\rfloor = \left\lfloor \log_2 \left(\frac{r+1}{2} \right) \right\rfloor = n-1$$

Otherwise, $r = 2^{n-1} + d$ where $0 \leq d < 2^{n-1} - 1$. So $\frac{r+1}{2} = 2^{n-2} + \frac{d+1}{2}$, where $\frac{d+1}{2} < 2^{n-2}$. Therefore

$$\log_2 \left\lfloor \frac{r+1}{2} \right\rfloor = n - 2 + \varepsilon$$

where $0 \leq \varepsilon < 1$. So

$$\left\lfloor \log_2 \left\lfloor \frac{r+1}{2} \right\rfloor \right\rfloor = n-2$$

Meanwhile,

$$\left\lfloor \log_2 \left(\frac{r+1}{2} \right) \right\rfloor = n - 2 + \delta$$

where $0 < \delta < 1$. So

$$\left\lfloor \log_2 \left(\frac{r+1}{2} \right) \right\rfloor = n-2$$

So

$$\left\lfloor \log_2 \left\lfloor \frac{r+1}{2} \right\rfloor \right\rfloor = \left\lfloor \log_2 \left(\frac{r+1}{2} \right) \right\rfloor = n-2$$

This compeletes the proof, because we have handled all possible values of r.