CLAUSAL INTUITIONISTIC LOGIC
I. FIXED-POINT SEMANTICS

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Since the advent of Horn-clause logic programming in the mid 1970's, there have been numerous attempts to extend the expressive power of Horn-clause logic while preserving some of its attractive computational properties. This article, the first of a pair, presents a clausal language that extends Horn-clause logic by adding negations and embedded implications to the right-hand side of a rule, and interpreting these new rules intuitionistically, in a set of partial models. The resulting system is shown to have a fixed-point semantics that generalizes the van Emden–Kowalski semantics for Horn clauses.

I. INTRODUCTION

Since the advent of Horn-clause logic programming in the mid 1970s, there have been numerous attempts to extend the expressive power of Horn-clause logic while preserving some of its attractive computational properties. Many of these efforts have focused on negation: Thus, for reasons of computational efficiency, the most common extension of Horn-clause logic uses the “negation-as-failure” rule [4], although there have been some recent attempts to implement a true negation rule in a computationally efficient manner [24]. Another extension, much less common, would permit an implication to appear “embedded” as the antecedent of another implication: for examples, see [13]. All of these extensions could be translated into full first-order logic, of course, with the desired inferences computed using a general-purpose theorem prover [2], but experience has shown that such an approach would be hopelessly inefficient in many realistic situations. The problem, then, is to strike an appropriate balance between expressive power and computational tractability.
In this article, we will develop an extended clausal language for logic programming that permits the appearance of both negations and embedded implications on the right-hand side of a rule, and yet preserves some of the computational properties of Horn-clause logic. As an example of the kinds of expressions we will analyse, consider the following:

\[ \text{UnownedProperty}(x) \leftarrow \neg \text{OwnActorProperty}(y, x), \]  
\[ \text{UnownedProperty}(x) \Rightarrow \neg \text{OwnActorProperty}(y, x). \]

The expression (1) should be read: “If no actor \( y \) owns property \( x \), then \( x \) is unowned property”; and the expression (2) should be read in the same way, but with the implication reversed: “If \( x \) is unowned property, then no actor \( y \) owns property \( x \).” Taken together, these two expressions provide a reasonable definition of “unowned property.” Note that the free variable \( y \) on the right hand side of each expression is given an implicit universal quantification, with its scope extending just outside of the negation sign, so that the translation of the first expression into standard notation would be

\[ (\forall y) [\text{UnownedProperty}(x) \leftarrow (\forall y) \neg \text{OwnActorProperty}(y, x)] \]

and the translation of the second expression into standard notation would be

\[ (\forall y) [\text{UnownedProperty}(x) \Rightarrow (\forall y) \neg \text{OwnActorProperty}(y, x)]. \]

When the universal quantifier is written in this way, the connection between negations and embedded implications becomes clear. Consider the following examples, which were discussed by John McCarthy in [15]:

\[ \text{SterileContainer}(x) \leftarrow [\text{InsideBugContainer}(y, x) \Rightarrow \text{DeadBug}(y)], \]  
\[ \text{SterileContainer}(x) \Rightarrow [\text{InsideBugContainer}(y, x) \Rightarrow \text{DeadBug}(y)]. \]

Using the same conventions as before, the expression (3) should be read: “If every bug \( y \) inside container \( x \) is a dead bug, then \( x \) is a sterile container”; and the expression (4) should be read: “If \( x \) is a sterile container, then every bug \( y \) inside container \( x \) is a dead bug.” Again, the free variable \( y \) on the right-hand side of each expression is given an implicit universal quantification, with its scope extending just outside of the embedded implications, so that the translation of (3) into standard notation would be

\[ (\forall y) [\text{SterileContainer}(x) \leftarrow (\forall y) [\text{InsideBugContainer}(y, x) \Rightarrow \text{DeadBug}(y)]] \]

and the translation of (4) would be:

\[ (\forall y) [\text{SterileContainer}(x) \Rightarrow (\forall y) [\text{InsideBugContainer}(y, x) \Rightarrow \text{DeadBug}(y)]] \]

The predicates in these examples are intended to suggest the use of a many-sorted language, with basic sorts for actors, properties, bugs, and containers. In fact, although we will not develop this point here, our system generalizes easily to such a language.

Now how are we to interpret these expressions, and draw inferences from them? Imagine, first, that we live in a world of positive atomic formulae and universally quantified Horn rules. We can thus observe particular atomic facts, such as \( \text{OwnActorProperty}(\text{Melvin, BlackAcre}) \) and \( \text{InsideBugContainer}(\text{Clyde, PetriDish}) \), and we can infer similar atomic facts from the application of the rules. Suppose we
want to determine whether or not "BlackAcre" is "unowned property". Using the negation-as-failure rule, we would check every actor and every ownership relation in the known world to determine whether any of these actors own Blackacre, and if not, we would conclude that "Blackacre is unowned property." This rule makes sense if we can rely on the "closed-world assumption" [22], but otherwise it will lead to a fallible inference. In the present article, however, we will assume that our observations are never complete, and that our view of the world can always be augmented by the discovery of new actors, and new bugs, and new relationships among them. We thus specifically reject the closed-world assumption. But if we only have partial information about the state of the world, at all times, how can we ever conclude with confidence that "BlackAcre is unowned property"? And how can we ever conclude with confidence that a particular "PetriDish" is a "SterileContainer"?

To see the alternative, suppose there is a rule stating that "y owns property x" only if "y has registered property x," and suppose we have been told authoritatively by the official registrar that "BlackAcre is unregistered property." We would then be justified in concluding that "no actor y has registered BlackAcre," and therefore "no actor y owns BlackAcre," and therefore "BlackAcre is unowned property." In this case, no matter how many additional actors turn up in our future observations of the state of the world, none of them could own BlackAcre. A similar analysis applies in the case of the sterile container. Since we have rejected the closed-world assumption, we cannot conclude that a particular Petri dish is sterile simply by examining each bug in the dish, one at a time, to see if it is dead. However, if we know that bugs are killed by prolonged heating, and if we know that anything inside a container is heated whenever the container is heated, and if we know that this particular Petri dish has been heated for at least five minutes, then we would be justified in concluding that the Petri dish is a sterile container. In this case, no matter how many additional bugs turn up in our future observations of the state of the world, every bug inside the container must be dead. The general point is this: We can draw conclusions about the right-hand side of a rule in the form (1) or (3) only if there exist rules in the form (2) or (4), or some equivalent Horn rules, that constrain the ways that our initial observation of the state of the world can be extended to a more complete state.

This article will develop the preceding point of view in formal detail, and analyze the properties of the resulting system of logic. To define the meaning of the rules (1)–(4) we will work with a set of partial models, each one of which represents a possible completion of our initial observation of the state of the world. The result is an intuitionistic semantics for negations and embedded implications [14,5], with the set of partial models corresponding to the set of possible worlds in a standard Kripke model. We will then consider an arbitrary set $\mathcal{R}$ consisting of Horn clauses, plus negation rules and embedded implication rules. It is well known [26,1] that there exists a simple fixed-point semantics for Horn clauses, and that a set of Horn clauses has a unique minimal model. We will show that a similar result holds for a set $\mathcal{R}$ that includes negations and embedded implications, as long as these additional rules are interpreted intuitionistically. In particular, we will construct a fixed-point operator for negations and embedded implications (see Theorem 3.7 below), and we will use this operator to establish the existence of a unique minimal model for the set of rules $\mathcal{R}$ (see Theorem 3.15 below). This appears to be a significant result: If the rules (1)–(4) are interpreted classically, it is clear that no
such minimal model exists. From a semantic perspective, therefore, our system of 
clausal intuitionistic logic turns out to be a natural generalization of Horn-clause 
logic.

To explore this relationship further, we will also investigate in Paper II a 
proposed proof procedure for clausal intuitionistic logic which is itself a natural 
generalization of Horn clause refutation procedures. One consequence of the 
existence of a unique minimal model for a set of rules $\mathcal{R}$ is that a query ‘$P(x)$’ has 
a definite answer substitution for the variable ‘$x$’ whenever ‘$(\exists x)P(x)$’ is entailed 
by $\mathcal{R}$. Our proof procedure thus resembles the standard Horn-clause proof proce-
dure in this respect: It begins with a formula ‘$P(x)$’ at the top node of a refutation 
tree, and, if it is successful, it concludes by returning an answer substitution for the 
variable ‘$x$’. But the refutation tree in our proof appears inside a structure called an 
initial tableau, and whenever the proof procedure encounters a negation rule in the 
form of (1), or an embedded implication rule in the form of (3), it creates a new 
structure called an auxiliary tableau which contains additional formulae and ad-
ditional refutation trees. We will show that this procedure is sound (Theorem 4.3) 
and complete (Theorem 4.6) with respect to our fixed point semantics, so that it 
faithfully computes the answers that are entailed by an intuitionistic interpretation 
of the negation and embedded implication rules. Obviously, the proof procedure 
would not be complete if these rules were interpreted classically.

Although the ultimate goal of this work is to develop a practical proof procedure 
for a system of rules including negations and embedded implications, questions of 
computational complexity are beyond the scope of the present article. Informally 
speaking, our proof procedure is somewhat more complex than Horn-clause proof 
procedures, since it uses a set of refutation trees in place of a single refutation tree; 
but it shares with Horn-clause logic programming a top-down goal-directed search 
strategy, and this suggests the possibility of building an efficient interpreter for our 
language. We will identify in Paper II one situation in which the nondeterminism of 
our proof procedure may lead to serious inefficiencies, and we will propose a 
plausible search heuristic for this case (see Section 4.6). However, our analysis of 
these issues is still incomplete. Thus, any statement about the existence of a 
practical proof procedure for clausal intuitionistic logic should be interpreted as a 
mere conjecture, to be validated or invalidated by subsequent research.

The idea of using an intuitionistic semantics for the set of rules (1)–(4) was first 
presented, in an abbreviated form, in [16]. That paper was also concerned with a 
broader topic: the relationship between intuitionistic logic and nonmonotonic 
reasoning. This broader topic is beyond the scope of the present article, but it will 
be treated in detail in a forthcoming work [17], which is the complete version of [16]. 
In [16], the construction of a fixed-point semantics for negations and embedded 
implications was outlined briefly, but the full details were not presented. Recently, 
Dale Miller has arrived independently at a similar result [18] using a slightly 
different construction. The work of Dov Gabbay and his colleagues on N-PROLOG 
and ON-PROLOG [6–8] is also closely related. Somewhat more remote, but still 
related, is the work of Peter Patel-Schneider on a decidable variant of first-order 
relevance logic [21]. Underlying all of this research there is a common theme: the 
hope is that we can weaken classical logic just enough to obtain a computationally 
tractable system without sacrificing too much expressive power in the language 
itself. We will discuss this related work in Section 5, which will appear in Paper II.
The section of the article immediately following is essentially a review of the standard fixed-point semantics for Horn-clause logic programming [26,1], adapted slightly to provide the foundation we need to develop the semantics of negations and embedded implications. The reader who is familiar with this material could easily skim through this section, noting only the points that diverge from the standard treatment. Section 3 then develops the semantics for the full set of rules \( \mathcal{R} \), including the fixed-point results. In Paper II, Section 4 will introduce the proof procedures, with examples, and then present the details of the soundness and completeness theorems. Finally, Section 6 will outline several extensions of the system that are left open for future work.

2. LOGIC PROGRAMMING IN A SYSTEM OF PARTIAL MODELS

In this section, we will review the standard fixed-point semantics for Horn-clause logic programming due to van Emden and Kowalski [26] and Apt and van Emden [1], but with a few minor variations to conform to the ideas developed later in the article. The basic position here is that logic programming should be viewed as a particular form of computation in a system of partial models. We assume that we have been given an initial observation of the state of the world, plus a set of rules that the complete state of the world must satisfy, and our task is to compute the answer to a query about certain aspects of the world that have not yet been directly observed. It will be seen that this is, indeed, a reasonable way to view standard logic programming, but more importantly it will turn out that this point of view is essential for an understanding of intuitionistic negations and embedded implications. In order to incorporate these intuitionistic rules into our system, it is necessary to modify slightly the usual interpretation of Horn clauses, but we will demonstrate in this section that the standard fixed-point results for Horn-clause logic are unaffected by these modifications. In Paper II, these results for Horn clauses will then play a crucial role in establishing the soundness (Theorem 4.3) and completeness (Theorem 4.6) of the tableau proof procedures for the full system of clausal intuitionistic logic.

We begin by defining the universe of discourse. In standard logic programming, this is taken to be the Herbrand universe, and the definition adopted here is almost identical. The only difference lies in the treatment of individual constants. An individual constant is usually defined as a function of zero arguments, and since the set of clauses in a logic program is always finite, this means that there can only be finitely many individual constants in the universe. But in our system, based on partial models, we imagine that we can always observe a more complete state of the world in which new individual constants, but no new function symbols, can appear. We must, therefore, make a fundamental distinction between individuals and function symbols. Let us then assume that our language consists of the following:

- a denumerable set of individuals: \( a, b, c, a_1, b_1, c_1, \ldots \);
- a denumerable set of variables: \( x, y, z, x_1, y_1, z_1, \ldots \);
- a finite set of function symbols: \( f, g, h, f_1, g_1, h_1, \ldots, f_m, g_m, h_m \); and
- a finite set of predicate symbols: \( P, Q, R, P_1, Q_1, R_1, \ldots, P_n, Q_n, R_n \).
The basic definitions are then standard. A \textit{term} is either an individual, or a variable, or an expression $f(t_1, \ldots, t_j)$ where $f$ is a function symbol and each $t_1, \ldots, t_j$ is a term. A term that contains no variables is called a \textit{ground term}. An \textit{atomic formula} is an expression $P(t_1, \ldots, t_k)$ where $P$ is a predicate symbol and each $t_1, \ldots, t_k$ is a term. An atomic formula that contains no variables is called a \textit{ground atomic formula}, or sometimes simply a \textit{ground instance}. Finally, we define $U$ to be the set of all ground terms constructible out of the given sets of individuals and function symbols, and we define $H$ to be the set of all ground atomic formulae constructible from the given set of predicates using only the terms in $U$. Intuitively, $U$ is the set of all possible objects that we can observe in our universe, and $H$ is the set of all possible relationships that can hold between these objects.

Now, in the absence of any further constraints, it is obvious that the complete state of the world could be any subset of $H$. This leads us naturally to a consideration of the set of all possible worlds, absent any constraints, which we will represent in the usual way by the powerset lattice over $H$. Formally, let $B = \mathcal{P}(H)$, and consider the complete lattice $[B, \sqsubseteq]$ defined on the set $B$ by taking the partial order relation `\sqsubseteq` to be ordinary set inclusion. We will generally refer to the elements of $B$ as \textit{substates}, to suggest that they may only be subsets of the complete state of the world, and we will denote them by an italic `s` with or without subscripts. We will denote the \textit{join} of two substates $s_1$ and $s_2$ by $s_1 \sqcup s_2$, and we will denote the \textit{meet} by $s_1 \sqcap s_2$. Since the lattice $[B, \leq]$ is complete, every set $J \subseteq B$ will have a \textit{least upper bound} and a \textit{greatest lower bound} in $B$, which we will denote by $\sqcup J$ and $\sqcap J$ respectively. Throughout the remainder of this article, we will take this basic lattice $[B, \sqsubseteq]$ to be fixed.

In all interesting cases, of course, the set of all possible worlds is subject to various constraints, and this leads to a natural notion of inference. We can think about this from the point of view of an intelligent agent, roaming around the world, making a series of partial observations, and trying to infer the relationships that must hold in the complete, but unobserved, state of the world. Assume that the agent has been given an \textit{initial substate} $s_0$, which is nonempty, and a set of rules $\mathcal{R}$ that impose constraints on the ways that this initial substate can be extended. (Note that $s_0$ and $\mathcal{R}$ will always be finite, for all practical purposes, although it is mathematically permissible to consider initial substates and sets of rules that are infinite.) The goal of the agent is then to determine which additional relationships it will observe in the complete state of the world. A reasonable (but stringent) requirement is the following:

\textit{Definition 2.1.} A ground instance $A$ is \textit{entailed} by an initial substate $s_0$ and a given set of rules $\mathcal{R}$ if and only if $A$ is a member of every substate $s \geq s_0$ that satisfies the rules in $\mathcal{R}$.

More generally, let $P(x)$ be any atomic formula with predicate $P$ and free variables $x$. It is useful to define two different notions of entailment for the formula $(\exists x)P(x)$:

\textit{Definition 2.2.} The formula $(\exists x)P(x)$ is \textit{entailed} by an initial substate $s_0$ and a given set of rules $\mathcal{R}$ if and only if, for every substate $s \geq s_0$ that satisfies the rules in $\mathcal{R}$, there exists some ground substitution $\theta$ such that $P(x)\theta \in s$. 

Definition 2.3. The formula \((\exists x)P(x)\) is uniformly entailed by an initial substate \(s_0\) and a given set of rules \(\mathcal{R}\) if and only if there exists some ground substitution \(\theta\) such that, for every substate \(s \geq s_0\) that satisfies the rules in \(\mathcal{R}\), \(P(x)\theta \subseteq S\).

In the second case, of course, the query ‘\(P(x)\)?’ will have a definite answer substitution for the variables \(x\). We will adopt here the usual conventions for substitutions. In particular, we will let \(\theta_1 \leq \theta_2\) mean that ‘\(\theta_1\) is less specific than \(\theta_2\),’ which will be true if and only if there exists a substitution \(\theta\) such that \(\theta_1 \circ \theta = \theta_2\). Also, we will often say that \(\theta\) is a ground substitution for the variables \(x\), which means that \(\theta\) leaves all variables other than \(x\) unbound. Note that if \(\theta_1\) and \(\theta_2\) are both ground substitutions, then we can have \(\theta_1 \leq \theta_2\) only if all the components of \(\theta_1\) also appear in \(\theta_2\).

There is an ambiguity in the preceding definitions that must now be clarified: We have not said what it means for a substate \(s\) to “satisfy the rules in \(\mathcal{R}\).” This will depend on the particular form of the rules, and for a set of Horn clauses the definition is particularly simple. We will write the rules for Horn clauses in two variants, as follows:

\[
P(x) \leftarrow Q_1(x; y) \land Q_2(x; y) \land \cdots \land Q_n(x; y),
\]

\[
P(x) \leftarrow Q_1(x; y) \lor Q_2(x; y) \lor \cdots \lor Q_n(x; y).
\]

The first rule will be called a conjunctive rule, and abbreviated by \(P \leftarrow \land Q_j\), and the second rule will be called a disjunctive rule, and abbreviated by \(P \leftarrow \lor Q_j\). These two rules together will also be referred to as abstraction rules, since the left-hand side of such a rule often represents an abstract predicate and the right-hand side often represents the definitional expansion of the predicate. Because of some technical details discussed in Paper II (see the proof of Lemma 4.8), it turns out to be convenient to require that a predicate \(P\) appears at most once on the left-hand side of the abstraction rules, and for this reason it is necessary to include disjunctive rules \(P \leftarrow \lor Q_j\) in the set \(\mathcal{R}\). Clearly, the set of rules \(P \leftarrow \land Q_j\) and \(P \leftarrow \lor Q_j\) with unique predicates on the left-hand side has the same expressive power as the usual set of rules \(P \leftarrow \land Q_j\) without this restriction.

The definition of “satisfaction” for the rules \(P \leftarrow \land Q_j\) and \(P \leftarrow \lor Q_j\) is simple because it depends only on a single substate \(s\). Let \(U(s)\) be the set of all ground terms that are constructible using only the individual constants appearing in the substate \(s\). We will adopt the following definitions:

Definition 2.4. The substate \(s\) satisfies the conjunctive rule \(P \leftarrow \land Q_j\) if and only if the following condition is true: Let \(\theta\) be any ground substitution for the variables in \(P(x)\) restricted to the terms in \(U(s)\). If there exists a ground substitution \(\theta' \geq \theta\) such that \(Q_j(x; y)\theta' \in s\) for all \(j\), then \(P(x)\theta \subseteq s\).

The substate \(s\) satisfies the disjunctive rule \(P \leftarrow \lor Q_j\) if and only if the following condition is true: Let \(\theta\) be any ground substitution for the variables in \(P(x)\) restricted to the terms in \(U(s)\). If there exists a ground substitution \(\theta' \geq \theta\) such that \(Q_j(x; y)\theta' \in s\) for some \(j\), then \(P(x)\theta \subseteq s\).

The reader will note that these are not quite the standard definitions. If every free variable \(x\) on the left-hand side of the rule also appears on the right-hand side of the
rule, then these definitions are logically equivalent to the standard definitions. But consider the case of a conjunctive rule \( P(x) \Leftarrow \) in which the right-hand side is empty. According to the standard definition, \( P(x) \theta \) would be included in the substate \( s \) for any ground substitution \( \theta \) that binds the variables \( x \) to any arbitrary terms in \( U \). According to Definition 2.4, \( P(x) \theta \) would be included in the substate \( s \) only if \( \theta \) binds the variables \( x \) to the terms in \( U(s) \). Of course, any substate \( s \) that satisfies the rule \( P(x) \Leftarrow \) under the standard definition would also satisfy the rule \( P(x) \Leftarrow \) under Definition 2.4. The important point, though, is that a substate \( s \) can satisfy the rule \( P(x) \Leftarrow \) according to Definition 2.4 without containing all the ground terms that appear in all the possible completions of \( s \). This is consistent with our basic philosophy: We are computing in partial models, and we are interpreting the rules in \( \mathcal{R} \) intuitionistically. We would not, therefore, want to commit ourselves to the "truth" of \( P(x) \theta \) for some \( \theta \) that is not yet "known" to "exist" in the substate \( s \).

To analyze this point further, and to establish a connection with the full intuitionistic interpretation that will be developed in Section 3, we need to consider not just a single substate \( s \) that satisfies the rules \( P \Leftarrow \land Q_j \) and \( P = \lor Q_j \), but also the set of all substates that satisfy these rules. For this purpose, the fixed-point construction due to van Emde and Kowalski [26] and Apt and van Emde [1] provides the appropriate technical machinery. Let us first fix a particular set of rules \( \mathcal{R} \) of the form \( P \Leftarrow \land Q_j \) and \( P = \lor Q_j \). With these rules fixed, we define an abstraction transformation \( T_A \) from \( B \) into \( B \) as follows:

**Definition 2.5.** Let \( s \) be a substate in \( B \). Define \( A \in T_A(s) \) if and only if:

- for some conjunctive rule \( P \Leftarrow \land Q_j \) and for some ground substitution \( \theta \) restricted to \( U(s) \), \( A = P(x) \theta \) and \( Q_j(x; y) \theta \in s \) for all \( j \); or
- for some disjunctive rule \( P = \lor Q_j \) and for some ground substitution \( \theta \) restricted to \( U(s) \), \( A = P(x) \theta \) and \( Q_j(x; y) \theta \in s \) for some \( j \).

We note that \( T_A \) is monotonic: If \( s_1 \leq s_2 \), then \( T_A(s_1) \leq T_A(s_2) \). We then use \( T_A \) to construct the set of all substates that satisfy the rules \( P \Leftarrow \land Q_j \) and \( P = \lor Q_j \), as follows:

**Lemma 2.6.** Let \( G = \{ s | s_0 \sqcup T_A(s) \leq s \} \). Then \( G \) is the set of all substates \( s \geq s_0 \) that satisfy the rules \( P \Leftarrow \land Q_j \) and \( P = \lor Q_j \).

**Proof.** Obvious, from a comparison of Definition 2.4 with Definition 2.5. □

Intuitively, the set \( G \) tells us all the possible ways that we can complete the description of the world observed at \( s_0 \), without violating the constraints in \( P = \land Q_j \) and \( P = \lor Q_j \). And \( G \) has another important property, as shown by the following:

**Lemma 2.7.** \( J \subseteq G \rightarrow \bigcap J \subseteq G \).

**Proof.** We will actually prove a slightly stronger result. Let \( T \) be any monotonic transformation from \( B \) into \( B \), and let \( G = \{ s | T(s) \leq s \} \). Consider any \( J \subseteq G \). By the definition of a greatest lower bound, \( \bigcap J \leq s \) for all \( s \in J \), and thus, by the monotonicity of \( T \), \( T(\bigcap J) \leq T(s) \) for all \( s \in J \). Now, by the definition of \( G \), if
FIGURE 1. The set of substates that satisfy $P \Leftarrow \land Q_j$ and $P \Leftarrow \lor Q_j$.

$s \in J \subseteq G$, it follows that

$$T(\cap J) \leq T(s) \leq s,$$

which means that $T(\cap J)$ is a lower bound of $J$. Since $\cap J$ is the greatest lower bound of $J$, it follows that $T(\cap J) \leq \cap J$. Thus, by the definition of $G$ again, $\cap J \in G$. To specialize this result to the present case, we simply take $T(s) = s_0 \cup T_A(s)$.

This result is generally referred to as the model intersection property for the rules $P \Leftarrow \land Q_j$ and $P \Leftarrow \lor Q_j$, since it asserts that the intersection of any set of substates that satisfy these rules is itself a substate that satisfies the rules. In particular, if we consider the substate $g^* = \cap G$, it follows from Lemma 2.7 that $g^* \in G$. The situation is illustrated in Figure 1.

It is now easy to see that entailment according to Definition 2.2 implies uniform entailment according to Definition 2.3 whenever the rules $\mathcal{R}$ take the form $P \Leftarrow \land Q_j$ and $P \Leftarrow \lor Q_j$. For suppose that $(\exists x)P(x)$ is entailed by the initial substate $s_0$ and the rules $\mathcal{R}$. Then, since the substate $g^* \in G$ satisfies the rules $\mathcal{R}$, there must exist some ground substitution $\theta$ such that $P(x)\theta \in g^*$. However, since $g^*$ is also the greatest lower bound of $G$, it follows that $P(x)\theta \in s$ for all $s \in G$. Thus $(\exists x)P(x)$ is uniformly entailed by $s_0$ and $\mathcal{R}$. The critical fact here is the existence of a unique minimal substate, namely $g^*$, which includes all the ground instances in the initial substate $s_0$ and which satisfies all the rules $\mathcal{R}$ in the form $P \Leftarrow \land Q_j$ and $P \Leftarrow \lor Q_j$. The substate $g^*$ can also be characterized as the least fixed point of the transformation $T(s) = s_0 \cup T_A(s)$, by virtue of the following theorem [25]:

**Theorem 2.8 (Knaster-Tarski fixed-point theorem).** Let $T$ be a monotonic operator on a complete lattice $[B, \leq]$. Let $P$ be the set of all fixed points of $T$. Then $P$ is nonempty, and the system $[P, \leq]$ is itself a complete lattice. In particular,

$$\bigcup P = \bigcup \{s | s \leq T(s)\} \in P$$

and

$$\bigcap P = \bigcap \{s | T(s) \leq s\} \in P.$$
It will be helpful at this point to summarize these results in a single theorem, which is an adaptation of one of the principal theorems in [26]:

**Theorem 2.9 (van Emden and Kowalski).** Let $s_0$ be a (possibly infinite) initial substate in $B$. Let $\mathcal{R}$ be a (possibly infinite) set of rules of the form $P \Leftarrow \Lambda Q_j$ and $P \Leftarrow \Lambda Q_k$, and let $T_\Lambda$ be the transformation associated with $\mathcal{R}$ by Definition 2.5. Define

$$g^* = \bigcap G = \bigcap \{s | s_0 \sqcup T_\Lambda(s) \leq s\}.$$ 

Then:

1. $g^*$ is the unique minimal substate containing $s_0$ that satisfies the rules in $\mathcal{R}$, and
2. $g^*$ is the least fixed point of the transformation $T(s) = s_0 \sqcup T_\Lambda(s)$.

Notice that we have stated this theorem in two parts. The first part, on the existence of a unique minimal substate, depends only on Lemma 2.7 and the model intersection property, and it leads to the following corollary:

**Corollary 2.10.** If $(\exists x)P(x)$ is entailed by $s_0$ and $\mathcal{R}$, then $(\exists x)P(x)$ is uniformly entailed by $s_0$ and $\mathcal{R}$.

The second part depends on the Knaster-Tarski fixed-point theorem, but it does not play a significant role in our analysis until we come to the proof of the completeness theorem for Horn-clause refutation (Theorem 2.14), where the least-fixed-point property becomes an essential ingredient in the proof. Even there, it is not sufficient for the proof of the completeness theorem that $g^*$ is the least fixed point of the transformation $T(s)$. For this it is necessary to establish an additional property of $T$: continuity.

We will say that a transformation $T$ on a complete lattice $[B, \sqcup, \sqcap]$ is *join-continuous* if, for every ascending chain $r_0 \leq r_1 \leq r_2 \leq \cdots$ in $B$, the following equality holds:

$$T(\sqcup \{r_k | k < \omega\}) = \sqcup \{T(r_k) | k < \omega\}.$$ 

The proof that $T(s) = s_0 \sqcup T_\Lambda(s)$ is join-continuous has a few additional steps in the present case, but it is otherwise identical to the proof in the standard treatment of Horn-clause logic [26,1]:

**Lemma 2.11.** The transformation $T(s) = s_0 \sqcup T_\Lambda(s)$ is join-continuous.

**Proof.** Let $C = \{r_k | k < \omega\}$ be any ascending chain of substates in $B$. We need to show that $T(\sqcup C) = \sqcup \{T(r_k) | k < \omega\}$. The inequality in one direction holds for any monotonic $T$ by the following simple argument: For any $r_k \in C$, $r_k \leq \sqcup C$ and therefore $T(r_k) \leq T(\sqcup C)$. But this means that $T(\sqcup C)$ is an upper bound on the set $\{T(r_k) | k < \omega\}$, from which it follows that

$$T(\sqcup C) \geq \sqcup \{T(r_k) | k < \omega\}.$$ 

The inequality in the opposite direction holds because of the definition of $T_\Lambda$. Let $A$ be any ground instance in $T(\sqcup C) = S_0 \sqcup T_\Lambda(\sqcup C)$. If $A \in S_0$ the inequality follows immediately, so assume that $A \in T_\Lambda(\sqcup C)$ by virtue of some conjunctive
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rule $P = Q \lor R$. By Definition 2.5, there exists some ground substitution $\theta$ restricted to $U(\square C)$ such that $A = P(x; \theta)$ and $Q_j(x; y; \theta) \in \square C$ for a finite number of indices $j$. But since $C$ is just an ascending chain of substates, this means that there must exist some substate $r_k \in S$ such that $Q_j(x; y; \theta) \in r_k$ for each $j$. If the ground terms in $\theta$ are included in $U(r_k)$, then the conditions of Definition 2.5 are satisfied for the substate $r_k$, and it follows that $A \in T_A(r_k)$. Otherwise, since there are only a finite number of individual constants in $\theta$, and since each of these constants appears in some ground atomic formula in $\square C$, there must exist some substate $r_{k+n}$ further up the chain such that $U(r_{k+n})$ includes all the ground terms in $\theta$. In this case, the conditions of Definition 2.5 are satisfied for the substate $r_{k+n}$, and it follows that $A \in T_A(r_{k+n})$. A similar analysis applies if $A \in T_A(\square C)$ by virtue of some disjunctive rule $P \lor Q$. We have thus shown that

$$T(\square C) \subseteq \bigsqcup \{ T(r_k) | k < \omega \}$$

as required. □

We now consider a sequence of transformations $T, T^2, T^3, \ldots$, applied to the empty substate $\emptyset$, and we prove the following result:

**Theorem 2.12.** $g^*_k = \bigsqcup \{ T^k(\emptyset) | k < \omega \}$.

**Proof.** Let $C = \{ T^k(\emptyset) | k < \omega \}$, and note that $\emptyset \leq T(\emptyset) \leq T^2(\emptyset) \leq \cdots$ is an ascending chain of substates in $B$. Thus

$$\square C = \bigsqcup \{ T^k(\emptyset) | k < \omega \}$$

$$= \bigsqcup \{ T(T^k(\emptyset)) | k < \omega \}$$

by virtue of Lemma 2.11, which means that $\square C$ is a fixed point of $T$. Since $g^*$ is the least fixed point of $T$, we know that $g^* \leq \square C$, and we need only show that $g^* \geq \square C$ to complete the proof of the theorem. But clearly $g^* \geq \emptyset$, and thus $g^* = T(g^*) \geq T(\emptyset)$, and thus by induction $g^* \geq T^k(\emptyset)$ for all $k$. So it follows that $g^* \geq \square C$. □

We note an immediate corollary:

**Corollary 2.13.** If $A \in g^*$, then all the individual constants in $A$ appear somewhere in the initial substate $s_0$, or in the rules $P = Q \lor R$, or $P = Q \lor R$.

We will use this fact in Section 4.5 in Paper II.

It is well known that the equality in Theorem 2.12 leads to a simple constructive proof of the completeness theorem for Horn-clause refutation [1]. However, there are several possible versions of Horn-clause refutation. We will use And/Or refutation trees throughout the present article [19, 20], instead of SLD refutation trees [1], and this requires us to form the unifying composition of the substitutions computed along the branches of the tree [23, 3]. The procedure is illustrated in Figure 2. In an SLD refutation procedure, which is standard in logic programming, the most general unifier $\sigma_2$ would be computed at the tip of the lower left branch of this tree, and then applied to the formula $'Q_2(x; y; \sigma_2)'$ in the lower right branch
FIGURE 2. An 'And/Or' refutation tree. Answer substitution: \( \sigma_1 \odot \sigma_2 \odot (\sigma_3 \oplus \sigma_4) \).

of the tree before the final unification with \( P_4(b) \) would be attempted. In an And/Or refutation procedure, however, the most general unifiers \( \sigma_3 \) and \( \sigma_4 \) would be computed separately along each branch, and the answer substitution would then be computed by forming the unifying composition, denoted here by \( \sigma_3 \oplus \sigma_4 \), along all of the “And” branches. It turns out that And/Or refutation trees are more convenient for the analysis of the tableau proof procedures that we will introduce in Section 4, but this is not an essential feature of our approach. In fact, in Section 4.6, after establishing the soundness and completeness theorems for a tableau proof procedure using And/Or refutation trees, we will show how to convert this into a tableau proof procedure using SLD refutation trees, if desired.

We will now state and prove the completeness theorem for Horn-clause refutation, assuming that all proofs are constructed from And/Or refutation trees. First, some terminology: If a refutation tree for the query \( P(x) \) has an answer substitution \( \sigma \) at its topmost node, we will say that the refutation tree for \( P(x) \) is closed with answer substitution \( \sigma \). If \( \sigma \) is any answer substitution, then the restriction of \( \sigma \) to the set of variables \( x \) will be denoted by \( \sigma(x) \). With this notation, we have the following result:

**Theorem 2.14 (Completeness).** Let \( s_0 \) be a (possibly infinite) initial substate in \( B \), and let \( \mathcal{R} \) be a (possibly infinite) set of rules of the form \( P \Leftarrow \bigwedge Q_j \) and \( P \Leftarrow \bigvee Q_j \). Assume that \( (\exists x) P(x) \) is uniformly entailed by \( s_0 \) and \( \mathcal{R} \) with ground substitution \( \theta \). Then there exists a closed refutation tree for \( P(x) \) with answer substitution \( \sigma \) such that \( \sigma(x) \leq \theta \).
PROOF. The proof is similar to the proof of Theorem 5.6 in [1]. If \((\exists x)P(x)\) is
uniformly entailed by \(s_0\) and \(R\) with ground substitution \(\theta\), then
\[
P(x)\theta \in g^* = \bigsqcup \{T^k(\theta) \mid k < \omega\}
\]
by Theorem 2.9 and Theorem 2.12. This means that \(P(x)\theta \in \bigsqcup \{T^k(\theta) \mid k \leq n\}\)
for some finite \(n\). It is therefore sufficient to prove the following proposition: For every
\(n < \omega\) and for every atomic formula \(P_0(x_0)\), if \(P_0(x_0)\theta \in \bigsqcup \{T^k(\theta) \mid k \leq n\}\)
for some ground substitution \(\theta\), then there exists a closed refutation tree for \(P_0(x_0)\) with
an answer substitution \(\sigma\) such that \(\sigma(x_0) \leq \theta\).

The proof of the proposition is by induction on \(n\). For the case \(n = 1\), \(P_0(x_0)\theta \in T(\theta) = s_0 \sqcup T_A(\theta)\). But \(T_A(\theta) = \emptyset\) by Definition 2.5, even if there are rules \(P \in R\)
that have null antecedents. Thus \(P_0(x_0)\theta \in s_0\). Thus the refutation tree for \(P_0(x_0)\)
closes immediately with the answer substitution \(\sigma = \theta\).

Now assume that the proposition is true for \(n\), and assume that
\[
P_0(x_0)\theta \in \bigsqcup \{T^k(\theta) \mid k < n + 1\} = T(\bigsqcup \{T^k(\theta) \mid k < n\})\).
\]
If \(P_0(x_0)\theta \in s_0\), then \(\sigma = \theta\) is again the desired answer substitution. Otherwise,
\(P_0(x_0)\theta \in T_A(\bigsqcup \{T^k(\theta) \mid k < n\})\) and we can apply Definition 2.5 directly. Assume
that \(P_0(x_0)\theta\) is generated by a conjunctive rule in the form \(P \Leftarrow \bigwedge Q_j\), and note
that the proof for a rule in the form \(P \Leftarrow \bigvee Q_j\) follows as a special case. By
Definition 2.5 there exists a ground substitution \(\theta_1\) such that
\[
P_0(x_0)\theta = P_1(x_1)\theta_1,
\]
and
\[
Q_{j,1}(x_1; y_1)\theta_1 \in \bigsqcup \{T^k(\theta) \mid k \leq n\}
\]
for each \(j\). Since the variables in \(P_0(x_0)\) and \(P_1(x_1)\) are distinct, we can rewrite (7)
as \(P_0(x_0)\theta \circ \theta_1 = P_1(x_1)\theta \circ \theta_1\), from which it is apparent that \(P_0(x_0)\) and \(P_1(x_1)\)
have a most general unifier \(\sigma_1 \leq \theta \circ \theta_1\). We can now verify the following equalities:
\[
Q_{j,1}(x_1; y_1)\theta_1 = Q_{j,1}(x_1; y_1)\theta \circ \theta_1 = Q_{j,1}(x_1; y_1)\sigma_1 \circ \rho_1
\]
for some ground substitution \(\rho_1\),

and use them to rewrite the expressions in (8). Applying the induction hypothesis to
these rewritten expressions, we see that there exists a closed refutation tree for
\(Q_{j,1}(x_1; y_1)\sigma_1\) with some answer substitution \(\sigma_{j,2}\) such that \(\sigma_{j,2}(x_0; x_1; y_1) \leq \rho_1\)
for each \(j\). Since the substitutions \(\sigma_{j,2}\) have a common instance for their shared
variables, though, they must also have a unifying composition \(\sigma_2\). Thus the refuta-
tion tree beginning with \(P_0(x_0)\) closes with an answer substitution \(\sigma = \sigma_1 \circ \sigma_2\). It is
straightforward to verify that \(\sigma(x_0) \leq \theta\). This completes the proof of Theorem 2.14.

Notice the crucial importance of Lemma 2.11 and Theorem 2.12 in this proof:
Without the continuity of \(T\), established in Lemma 2.11, the first step of the proof
would not go through.

For the soundness theorem, it is necessary to introduce a minor qualification, as
shown by the following example: Suppose \(s_0 = \{Q(c)\}\), and suppose \(R\) contains
the single rule \(\{P(x) \Leftarrow \}\). Then the refutation tree beginning with \(\{P(a)\}\) would be
closed, but 'P(a)' would not be entailed by \( s_0 \) and \( \mathcal{R} \) according to Definition 2.1. Similarly, suppose \( \mathcal{R} \) also contains the rule 'Q(x) \leftarrow P(a)' Then the refutation tree beginning with 'Q(b)' would be closed, but 'Q(b)' would not be entailed by \( s_0 \) and \( \mathcal{R} \) according to Definition 2.1. To correct this situation, we will require that the individual constants in the query P(x) and the rules \( \mathcal{R} \) also appear in the initial substate \( s_0 \). Note that an analogous requirement is necessary in the standard treatment of Horn-clause logic: see [26, p. 736] and [1, p. 847].

We will first prove the following lemma, which is itself a useful form of the soundness theorem:

**Lemma 2.15.** Let \( s_0 \) be a (possibly infinite) initial substate in \( \mathcal{B} \), and let \( \mathcal{R} \) be a (possibly infinite) set of rules of the form \( P \leftarrow \bigwedge Q_j \) and \( P \leftarrow \bigvee Q_j \). Assume that there exists a closed refutation tree for \( P(x) \) with answer substitution \( \sigma \). Then for every substate \( s \in \mathcal{G} \) that includes the individual constants in \( P(x) \) and \( \mathcal{R} \), and for every ground substitution \( \theta \) for the free variables in \( P(x) \) restricted to the terms in \( \mathcal{U}(s) \), it follows that \( P(x)\sigma \circ \theta \in s \).

**Proof.** The proof is similar to the proof of Theorem 5.1 in [1]. Assume that the query \( P(x) \) has unified at the top node of the refutation tree with some atomic formula \( P_i(x) \) from the left-hand side of a rule \( P \leftarrow \bigwedge Q_j \) or \( P \leftarrow \bigvee Q_j \), and let \( \sigma_i \) be the most general unifier at that node. We will say that the depth of this top node is \( 0 \). In general, for an arbitrary node \( \mathcal{N} \) of depth \( k > 0 \), we will assume that a formula \( Q_k(x; y)\sigma_k \) that was derived from the right-hand side of a rule \( P \leftarrow \bigwedge Q_j \) or \( P \leftarrow \bigvee Q_j \) and a formula \( P_{k+1}(x) \) from the left-hand side of such a rule have a most general unifier \( \sigma_{k+1} \) at the node \( \mathcal{N} \), and we will assume that the proof has returned an answer substitution \( \sigma_{k+1} \) to that node. Finally, we will assume that the node at the end of the longest branch in the refutation tree has depth \( n \). We will now prove the following proposition:

**Proposition 2.16.** Let \( Q_{n-i}(x; y)\sigma_{n-i} \) be the formula in a node \( \mathcal{N} \) of depth \( n - i \) in the refutation tree, and let \( \sigma'_{n-i+1} \) be the answer substitution returned to \( \mathcal{N} \). Then for every substate \( s \in \mathcal{G} \) that includes the individual constants in \( P(x) \) and \( \mathcal{R} \), and for every ground substitution \( \theta \) for the free variables in \( Q_{n-i}(x; y)\sigma_{n-i} \circ \sigma'_{n-i+1} \) restricted to the terms in \( \mathcal{U}(s) \), it follows that \( Q_{n-i}(x; y)\sigma_{n-i} \circ \sigma'_{n-i+1} \circ \theta \in s \).

The proof is by induction on \( i \). For the case \( i = 0 \), the node \( \mathcal{N} \) would be a terminal node, and the formula \( Q_n(x; y)\sigma_n \) would have unified either with a ground instance in \( s_0 \), or with a formula \( P_{n+1}(x) \) from the left-hand side of a rule \( P(x) \leftarrow \) with a null antecedent. In either case, the answer substitution \( \sigma'_{n+1} \) would be identical to the most general unifier \( \sigma_{n+1} \) at the node \( \mathcal{N} \). If the node \( \mathcal{N} \) terminates in a ground instance \( A \in s_0 \), then \( Q_n(x; y)\sigma_n \circ \sigma_{n+1} = A \) and Proposition 2.16 follows immediately. On the other hand, if the node \( \mathcal{N} \) terminates in a formula from the left-hand side of a rule \( P(x) \leftarrow \), then

\[
Q_n(x; y)\sigma_n \circ \sigma_{n+1} = P_{n+1}(x)\sigma_{n+1},
\]

and Proposition 2.16 follows from Definition 2.4 together with the fact that the substate \( s \) is a member of \( \mathcal{G} \). Note that the substitution \( \sigma_{n+1} \) might very well contain some of the individual constants appearing in the query \( P(x) \) or the rules \( \mathcal{R} \),
but we have stipulated that these constants are also included in s, and thus the restriction in Definition 2.4 causes no problems.

Now assume that Proposition 2.16 has been established for all nodes of depth \( n - i \), and consider a node \( \mathcal{N} \) of depth \( n - i - 1 \). If \( \mathcal{N} \) is a terminal node, then the preceding arguments apply directly. Otherwise, the formula \( Q_{n-i-1}(x; y)\sigma_{n-i-1} \) would have unified with a formula \( P_{n-i}(x) \) from the left-hand side of a rule \( P = \land Q_j \) or \( P = \lor Q_j \), so that

\[
Q_{n-i-1}(x; y)\sigma_{n-i-1} \circ \sigma_{n-1} = P_{n-i}(x)\sigma_{n-1}
\]

for some substitution \( \sigma_{n-1} \). In this case, the node \( \mathcal{N} \) would be extended to a set of nodes \( \{ \mathcal{N}' \} \) of depth \( n - i \) containing the formulae \( Q_{j,n-i}(x; y)\sigma_{n-i} \) along with the answer substitutions \( \sigma_{j,n-i+1} \), and the answer substitution \( \sigma_{n-i+1} \) at node \( \mathcal{N} \) would be equal to the composition of \( \sigma_{n-i} \) with the unifying composition of all the substitutions \( \sigma_{j,n-i+1} \). Suppose now that \( \theta \) is any ground substitution for the free variables in \( Q_{n-i-1}(x; y)\sigma_{n-i-1} \circ \sigma_{n-1} \), restricted to the terms in \( U(s) \). We need to show that \( Q_{n-i-1}(x; y)\sigma_{n-i-1} \circ \sigma_{n-1} \circ \theta \in s \). However, since each node \( \mathcal{N}' \) satisfies the induction hypothesis for some ground substitution \( \theta' \geq \theta \), it follows that

\[
Q_{j,n-i}(x; y)\sigma_{n-i} \circ \sigma_{j,n-i+1} \circ \theta' \in s
\]

for each \( j \), and then by Definition 2.4 it follows that

\[
P_{n-i}(x)\sigma_{n-i} \circ \theta \in s.
\]

We have thus established Proposition 2.16 for an arbitrary node \( \mathcal{N} \) of depth \( n - i - 1 \), and it follows that the proposition is true for all \( i \).

For the case \( i = n \), the formula in the node \( \mathcal{N} \) of depth 0 is simply the query \( P(x) \), and the answer substitution is \( \sigma \). This completes the proof of Lemma 2.15. \( \square \)

We will now convert Lemma 2.15 into a soundness theorem that parallels more closely the completeness theorem established above. Assume that the initial substate \( s_0 \) includes all the individual constants in \( P(x) \) and \( \mathcal{R} \). If \( P(x)\sigma \) is a ground atomic formula, then Lemma 2.15 tells us that \( P(x)\sigma \in g^* \). Thus \( (3x)P(x) \) is uniformly entailed by \( s_0 \) and \( \mathcal{R} \) with ground substitution \( \sigma(x) \). Otherwise, if \( \theta \) is any ground substitution that binds the free variables in \( P(x)\sigma \) to the ground terms in \( U(g^*) \), then Lemma 2.15 tells us that \( P(x)\sigma \circ \theta \in g^* \). Thus \( (3x)P(x) \) is uniformly entailed by \( s_0 \) and \( \mathcal{R} \) with ground substitution \( \sigma(x) \circ \theta \). We will restate this result as follows:

**Theorem 2.17 (Soundness).** Let \( s_0 \) be a (possibly infinite) initial substate in \( \mathcal{R} \), and let \( \mathcal{R} \) be a (possibly infinite) set of rules of the form \( P = \land Q_j \) and \( P = \lor Q_j \). Assume that the individual constants in \( P(x) \) and \( \mathcal{R} \) appear in \( s_0 \), and assume that there exists a closed refutation tree for \( P(x) \) with answer substitution \( \sigma \). Then \( (3x)P(x) \) is uniformly entailed by \( s_0 \) and \( \mathcal{R} \) with some ground substitution \( \theta \) such that \( \sigma(x) \leq \theta \).

Apt and van Emden pointed out in their original paper [1] that the soundness and completeness of Horn-clause refutation could, of course, be derived from the soundness and completeness of resolution in general, by a simple specialization. But the soundness of resolution in general only guarantees that \( (3x)P(x) \) is true, whereas Theorem 2.17 (soundness) tells us that \( P(x)\theta \) is true for some ground
substitution \( \theta \) such that \( \sigma(x) \leq \theta \), and this fact justifies the use of Horn-clause refutation as a computational mechanism. A similar point could be made about completeness. If we assume that \( (\exists x)P(x) \) is true, then Corollary 2.10 to Theorem 2.9 tells us that \( P(x) \theta \) is true for some ground substitution \( \theta \), and Theorem 2.14 (completeness) then tells us that Horn-clause refutation will produce an answer substitution \( \sigma \) such that \( \sigma(x) \leq \theta \). Again, we obtain more information from these two theorems than we do from the completeness of resolution in general. In addition, because of the existence of a fixed-point semantics and a continuous transformation \( T \), the proof of the completeness theorem for Horn-clause refutation is relatively simple.

Our objective in the remainder of this article is to show that a system of rules including negations and embedded implications has some of these same characteristics, as long as the new rules are interpreted intuitionistically.

3. NEGATIONS AND EMBEDDED IMPLICATIONS: SEMANTICS

In this section, we will develop an intuitionistic semantics for a set of rules including negations and embedded implications, following the pattern established in Section 2. It happens that the set \( G = \{ s \mid s_0 \cup T_A(s) \leq s \} \), constructed in Theorem 2.9, is itself an intuitionistic model for the rules \( P \wedge Q \) and \( P \vee Q \). In Section 3.1, we will generalize this fact in a natural way, and define the meaning of the negation and embedded implication rules for arbitrary sets \( J \subseteq B \). Then, in Section 3.2, we will construct a fixed-point operator for these rules, and we will establish the existence of a greatest fixed point \( K^* \) among all the sets \( J \subseteq B \). The principal theorem here, Theorem 3.7, bears a striking resemblance to Theorem 2.9, but a distinction emerges upon closer examination: The fixed-point operator for negations and embedded implications is not continuous. This fact will cause complications in the proof of the completeness theorem in Section 4, but they are not fatal complications, and we will close our discussion of the fixed-point semantics for clausal intuitionistic logic with a positive result: We will show in Section 3.3 that the set \( K^* \) possesses the model intersection property, just like the set \( G \) in the Horn-clause case. Finally, in Section 3.4, we will discuss three examples that illustrate various aspects of the fixed-point theory. In particular, we will discuss the counterexamples that demonstrate that the fixed-point operator for negations and embedded implications is noncontinuous.

3.1. An Intuitionistic Interpretation

Let us return to a consideration of the rules (1)–(4), which were analyzed informally, for an unowned property and a sterile container, in Section 1. The general form for the rules (1) and (2) is the following:

\[
\begin{align*}
P(x) &\iff \neg Q(x; y) \\
P(x) &\Rightarrow \neg Q(x; y)
\end{align*}
\]

We will call these negation rules and abbreviate them by \( P \leftarrow \neg Q \) and \( P \Rightarrow \neg Q \). The general form for the rules (3) and (4) is the following:

\[
\begin{align*}
P(x) &\iff [Q(x; y) \Rightarrow R(x; y)] \\
P(x) &\Rightarrow [Q(x; y) \Rightarrow R(x; y)]
\end{align*}
\]
We will call these *embedded implication rules* and abbreviate them by \( P \leftarrow [Q \Rightarrow R] \) and \( P \Rightarrow [Q \Rightarrow R] \). We will also refer to the rules (9) and (11) as *abstraction rules*, since the left-hand side of such a rule often represents an abstract predicate, and we will refer to the rules (10) and (12) as *expansion rules*, since the right-hand side of such a rule often represents the definitional expansion of a predicate. In all cases, the variables \( y \) will be given an implicit universal quantification with scope extending just outside the right-hand side of the rule. We will rely on the examples in Section 1 to convey an informal understanding of the intended interpretation of these rules.

Now how should we formalize this intended interpretation? The key idea, as suggested in Section 1, is to imagine all the possible ways that we can *complete* the description of the world observed at \( s_0 \). If our world is constrained only by rules of the form \( P \leftarrow \wedge Q_j \) and \( P \leftarrow \vee Q_j \), then the set \( G = \{ s \mid s_0 \cup T(s) \leq s \} \) tells us all the possible substates \( s \geq s_0 \). But suppose our world is also constrained by a rule \( P(x) \leftarrow \neg Q(x; y) \). This means that a future observation of \( P(x) \), for any instantiation of the variables \( x \), will preclude a subsequent observation of \( Q(x; y) \), for any instantiation of the variables \( y \). Suppose our world is further constrained by a rule \( P(x) \leftarrow [Q(x; y) \Rightarrow R(x; y)] \). This means that a future observation of \( P(x) \), for any instantiation of the variables \( x \), will force all subsequent observations of the state of the world to be constrained by the rule \( Q(x; y) \Rightarrow R(x; y) \). From these considerations, we see that the set of substates \( J \) that satisfies the additional rules \( P \Rightarrow \neg Q \) and \( P \Rightarrow [Q \Rightarrow R] \) is, typically, a proper subset of the set \( G \) that satisfies the rules \( P \leftarrow \wedge Q_j \) and \( P \leftarrow \vee Q_j \). Now consider a negation rule with the reverse implication: \( P(x) \leftarrow \neg Q(x; y) \). To apply this rule we have to identify *all possible substates* that satisfy the known constraints on our world, and we have to determine whether \( Q(x; y) \) is true in *any* of these substates. If \( Q(x; y) \) is *not* true, in *all* possible substates, for *all* instantiations of the variables \( y \), then the rule tell us that \( P(x) \) must itself be true. A similar analysis applies to the rule \( P(x) \leftarrow [Q(x; y) \Rightarrow R(x; y)] \).

In this case, we must determine whether the embedded implication \( Q(x; y) \Rightarrow R(x; y) \) on the right-hand side is true in *all possible substates* that satisfy the known constraints on our world. It is obvious, however, that there is a potential interaction between these two kinds of rules. The rules \( P \Rightarrow \neg Q \) and \( P \Rightarrow [Q \Rightarrow R] \) might restrict the set \( J \) of possible substates, which might then trigger the right-hand side of some of the rules \( P \leftarrow \neg Q \) and \( P \leftarrow [Q \Rightarrow R] \), which might then trigger further rules of the form \( P \Rightarrow \neg Q \) and \( P \Rightarrow [Q \Rightarrow R] \), and so on. To avoid this problem, we need to find the *largest* set \( J \subseteq B \) that satisfies *all the rules simultaneously*, if such a set exists. These considerations thus lead naturally to the construction of a fixed-point semantics for negations and embedded implications.

With this motivation, we can now proceed to the formal definitions. Let \( U(s) \) be the set of all ground terms that are constructible using only the individual constants appearing in the substate \( s \). Let \( J \) be any subset of \( B \). We first define the meaning of the negation rules \( P \leftarrow \neg Q \) and \( P \Rightarrow \neg Q \) relative to the set \( J \), as follows:

**Definition 3.1.** The substate \( s \in J \) satisfies the negation rule \( P \leftarrow \neg Q \) if and only if the following condition is true: Let \( \theta \) be any ground substitution for the variables in \( P(x) \) restricted to the terms in \( U(s) \). If \( Q(x; y)\theta \notin s' \) for all \( s' \geq s \) in \( J \) and for all ground substitutions \( \theta' \geq \theta \), then \( P(x)\theta \in s \).
The substate \( s \in J \) satisfies the negation rule \( P \Rightarrow \neg Q \) if and only if the following condition is true: Let \( \theta \) be any ground substitution for the variables in \( P(x) \) restricted to the terms in \( U(s) \). If \( P(x)\theta \in s \), then \( Q(x;y)\theta' \notin s' \) for all \( s' \geq s \) in \( J \) and for all ground substitutions \( \theta' \geq \theta \).

Similarly, we define the meaning of the embedded implication rules \( P\Leftarrow [Q \Rightarrow R] \) and \( P \Rightarrow [Q \Rightarrow R] \) as follows:

**Definition 3.2.** The substate \( s \in J \) satisfies the embedded implication rule \( P \Leftarrow [Q \Rightarrow R] \) if and only if the following condition is true: Let \( \theta \) be any ground substitution for the variables in \( P(x) \) restricted to the terms in \( U(s) \). If \( Q(x;y)\theta' \in s' \Rightarrow R(x;y)\theta' \in s' \) for all \( s' \geq s \) in \( J \) and for all ground substitutions \( \theta' \geq \theta \) restricted to the terms in \( U(s') \), then \( P(x)\theta \in s \).

The substate \( s \in J \) satisfies the embedded implication rule \( P \Rightarrow [Q \Rightarrow R] \) if and only if the following condition is true: Let \( \theta \) be any ground substitution for the variables in \( P(x) \) restricted to the terms in \( U(s) \). If \( P(x)\theta \in s \), then \( Q(x;y)\theta' \in s' \Rightarrow R(x;y)\theta' \in s' \) for all \( s' \geq s \) in \( J \) and for all ground substitutions \( \theta' \geq \theta \) restricted to the terms in \( U(s') \).

Although these definitions are relativized to an arbitrary set \( J \subseteq B \), we will now say that a set \( J \) itself satisfies one of these rules if every substate \( s \in J \) satisfies the rule according to the preceding definitions. Note that any such set \( J \) is essentially a Kripke model for the given rule, and thus our interpretation is an intuitionistic interpretation \([14,5]\) of the rules \((9)-(12)\).

Definitions 3.1 and 3.2 are given here for the most general forms of the negation and embedded implication rules, but we can simplify these rules in certain cases. It is convenient to separate the rules into two groups: the expansion rules \( P \Rightarrow \neg Q \) and \( P \Rightarrow [Q \Rightarrow R] \), and the abstraction rules \( P \Leftarrow \neg Q \) and \( P \Leftarrow [Q \Rightarrow R] \). We note, first, that the expansion rule \( P \Rightarrow [Q \Rightarrow R] \) is equivalent to the Horn clause \( R \Leftarrow P \land Q \), in the following sense:

**Lemma 3.3.** The set \( J \subseteq B \) satisfies the expansion rule

\[ P(x) \Rightarrow [Q(x;y) \Rightarrow R(x;y)] \]

if and only if every substate \( s \in J \) satisfies the Horn clause

\[ R(x;y) \Leftarrow P(x) \land Q(x;y). \]

**Proof.** Obvious, from a comparison of Definition 2.4 with Definition 3.2. \( \square \)

We can thus simplify our system by dropping all rules in the form \( P \Rightarrow [Q \Rightarrow R] \), and working only with the expansion rules \( P \Rightarrow \neg Q \). Later, in Section 4.6, in a discussion of the possible variations in our proof procedures, we will show how to add the expansion rules \( P \Rightarrow [Q \Rightarrow R] \) back to the proofs, if desired.

Now consider the abstraction rules \( P \Leftarrow \neg Q \) and \( P \Leftarrow [Q \Rightarrow R] \). We can simplify the form of the abstraction rules \( P \Leftarrow [Q \Rightarrow R] \) by assuming that all the variables \( x \) and \( y \) that appear in \( R(x;y) \) also appear in \( Q(x;y) \). To see this, suppose there is a rule

\[ P(x_1;x_2) \Leftarrow [Q(x_1;y_1) \Rightarrow R(x_1;x_2;y_1;y_2)] \]
that violates this assumption. We can then replace this rule with the rule

$$P(x_1; x_2) \leftarrow [Q'(x_1; x_2; y_1; y_2) \Rightarrow R(x_1; x_2; y_1; y_2)],$$

plus the two Horn clauses

$$Q'(x_1; x_2; y_1; y_2) \Rightarrow Q(x_1; y_1),$$
$$Q'(x_1; x_2; y_1; y_2) \Rightarrow Q(x_1; y_1),$$

where the new formula $Q'(x_1; x_2; y_1; y_2)$ is identical to the formula $Q(x_1; y_1)$ except that the predicate $Q'$ has additional arguments for the variables $x_2$ and $y_2$. From an inspection of Definition 2.4 and Definition 3.2, it is easy to see that these two systems of rules are equivalent. Similarly, we may assume that the variables $x$ on the left-hand side of the abstraction rules $P \equiv \neg Q$ and $P \equiv [Q \Rightarrow R]$ also appear on the right-hand side of these rules. These assumptions will simplify the analysis of the proof procedures that will be introduced in Paper II, to appear in the upcoming June issue. We note one consequence immediately: If, for every rule $P \equiv [Q \Rightarrow R]$, the $y$ variables in $R(x; y)$ also appear in $Q(x; y)$, then it becomes superfluous in Definition 3.2 to restrict the substitution $\theta'$ to the terms in $U(s')$. We will thus assume in our subsequent analysis that this restriction has been eliminated.

3.2. Fixed-Point Semantics

We will now consider a set of rules $\mathcal{R}$ that includes Horn clauses in the form $P \equiv \land Q_j$ and $P \equiv \lor Q_j$, expansion rules in the form $P \Rightarrow \neg Q$, and abstraction rules in the form $P \equiv \neg Q$ and $P \equiv [Q \Rightarrow R]$. Our task is to construct a fixed-point semantics for these rules.

We will work with the expansion rules first. It is clear from Definition 3.1 that some sets $J \subseteq B$ will satisfy the rules $P \Rightarrow \neg Q$, and some will not, and we obviously want to consider only those sets $J$ that do satisfy the given rules. As an analytical device, then, let us construct the (much larger) lattice $[\mathcal{P}(B), \subseteq]$, using the symbols $\cup$ and $\cap$ for the join and the meet, and also for the least upper bound and the greatest lower bound. We can then establish the following result:

**Lemma 3.4.** The set of all sets $J \subseteq G$ that satisfy the negation rules $P \Rightarrow \neg Q$ has a largest element in $[\mathcal{P}(B), \subseteq]$, which we will denote by $J_0$.

**Proof.** Define

$$J_0 = \cup\{J \mid J \subseteq G \text{ and } J \text{ satisfies all rules } P \Rightarrow \neg Q\}.$$

It is obvious that $J_0 \subseteq G$, and thus we need only show that $J_0$ satisfies the rules $P \Rightarrow \neg Q$. Let $s$ be any substate in $J_0$, and select one of the negation rules $P(x) \Rightarrow \neg Q(x; y)$ for consideration. Assume $P(x) \theta \in s$ for some $\theta$ restricted to $U(s)$, and consider any $s' \in J_0$ such that $s' \supseteq s$. By the definition of $J_0$, $s' \in J$ for some $J \subseteq G$ that satisfies all the negation rules $P \Rightarrow \neg Q$, and since $P(x) \theta \in s'$, it follows that $Q(x; y) \theta' \not\in s'$ for all $\theta' \supseteq \theta$. Thus $J_0$ satisfies the selected negation rule $P \Rightarrow \neg Q$ at $s$. Since the substate $s$ and the rule $P(x) \Rightarrow \neg Q(x; y)$ were arbitrary, however, we conclude that $J_0$ satisfies all rules in the form $P \Rightarrow \neg Q$. Clearly $J_0$ is the largest subset of $G$ for which this is true. □
This means that we need only consider the single set $J_0$ in our subsequent analysis of the rules $P \Leftarrow \lnot Q$, since every set that satisfies these rules will be contained in $J_0$.

The analysis of the abstraction rules $P \Leftarrow \lnot Q$ and $P \Leftarrow [Q \Rightarrow R]$ is slightly more complicated, but here we can mimic the approach taken in the Horn-clause case. Let us define a transformation $T_N$ from $\mathcal{P}(B)$ into $\mathcal{P}(B)$ as follows:

**Definition 3.5.** Let $J$ be a subset of $B$. Define $s \in T_N(J)$ if and only if $s \in J$ and for every ground instance $A$ such that

- for some negation rule $P \Leftarrow \lnot Q$ and for some ground substitution $\theta$ for the variables in $P(x)$ restricted to the terms in $U(s)$, $A = P(x)\theta$ and $Q(x; y)\theta' \in s'$ for all $s' > s$ in $J$ and for all ground substitutions $\theta' > \theta$, or

- for some embedded implication rule $P \Leftarrow [Q \Rightarrow R]$ and for some ground substitution $\theta$ for the variables in $P(x)$ restricted to the terms in $U(s)$, $A = P(x)\theta$ and $Q(x; y)\theta' \in s'$, $R(x; y)\theta' \in s'$ for all $s' \geq s$ in $J$ and for all ground substitutions $\theta' \geq \theta$,

it follows that $A \in s$.

Although this definition may seem complex, the idea is actually very simple. Basically, $T_N$ is a deletion operator: It examines the set $J$ above the substate $s$ to see if the right-hand side of any of the rules $P \Leftarrow \lnot Q$ or $P \Leftarrow [Q \Rightarrow R]$ is satisfied, and, if so, it deletes the substate $s$ unless the left-hand side of each of these rules is also satisfied. We can then show the following:

**Lemma 3.6.** $T_N$ is a monotonic transformation from $\mathcal{P}(B)$ into $\mathcal{P}(B)$.

**Proof.** Assume $J_1 \subseteq J_2$, and select any substate $s \in T_N(J_1)$. By Definition 3.5, $s \in J_1$, and thus $s \in J_2$, and if we can show that the remaining conditions of Definition 3.5 are satisfied as well, we will be able to show that $s \in T_N(J_2)$. Suppose there is a negation rule $P \Leftarrow \lnot Q$ and a ground substitution $\theta$ restricted to $U(s)$ such that $Q(x; y)\theta' \notin s'$ for all $s' \geq s$ in $J_2$ and for all $\theta' \geq \theta$. Then the same condition will hold for all $s' \geq s$ in $J_1$. Since $s \in T_N(J_1)$, though, it must be the case that $P(x)\theta \in s$. Since this same argument applies to any rule $P \Leftarrow \lnot Q$ or $P \Leftarrow [Q \Rightarrow R]$ involved in Definition 3.5, it follows that $s \in T_N(J_2)$. We have thus shown that $T_N(J_1) \subseteq T_N(J_2)$. $\square$

Since we now have a monotonic transformation on the complete lattice $[\mathcal{P}(B), \subseteq]$, we can apply the Knaster-Tarski fixed-point theorem [25] directly to construct a complete lattice of fixed points in $[\mathcal{P}(B), \subseteq]$. In this case, we will be interested in the *greatest* fixed point, rather than the least fixed point, but otherwise the correspondence to the fixed-point semantics of van Emden and Kowalski [26] and Apt and van Emden [1] is very close. In particular, the following theorem is closely analogous to Theorem 2.9:

**Theorem 3.7.** Let $s_0$ be a (possibly infinite) initial substate in $B$, and let $\mathcal{R}$ be a (possibly infinite) set of rules including negations and embedded implications. Let $J_0 \subseteq \mathcal{P}(B)$ be the largest set of substates $s \geq s_0$ that satisfies the rules $P \Leftarrow \lnot Q$,.
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Let $T_N$ be the transformation associated with the rules $P \leftarrow \neg Q$, and $P \Rightarrow \neg R$. Define

$$K^* = \bigcup \{ J | J \subseteq T_N(J) \cap J_0 \}.$$ 

Then:

1. $K^*$ is the largest set of substates $s \geq s_0$ that satisfies all the rules in $\mathcal{R}$, and

2. $K^*$ is the greatest fixed point of the transformation $T(J) = T_N(J) \cap J_0$.

**Proof.** Set $K^* = \bigcup \{ J | J \subseteq T(J) \} = \bigcup \mathcal{P}$ in Theorem 2.8. Then $K^*$ is the greatest fixed point of the transformation $T(J) = T_N(J) \cap J_0$. In other words, $K^* = T_N(K^*) \cap J_0$. Now since $K^* \subseteq J_0$, it follows that $K^*$ satisfies the rules $P \leftarrow \land Q$, $P \leftarrow \lor Q$, and $P \Rightarrow \neg Q$. And since $K^* \subseteq T_N(K^*)$, it follows from a comparison of Definitions 3.1 and 3.2 with Definition 3.5 that $K^*$ satisfies the rules $P \equiv Q$ and $P \equiv [Q \Rightarrow R]$. Clearly, $K^*$ is the largest set of substates $s \geq s_0$ with these properties. 

Note that the first part of this theorem could be based on a model union property, analogous to Lemma 2.7, without reference to the fixed-point construction in the second part of the theorem. However, the proof of Lemma 2.7 and the analogous proof of the model union property are similar to the first part of the proof of the Knaster-Tarski fixed-point theorem [25], and there appears to be no reason to separate these two results in this case.

Let us now return to the problem of entailment for negations and embedded implications, which we began to discuss in Section 3.1. Recall that we were looking for the largest set in $\mathcal{P}(B)$ that satisfies all the rules in $\mathcal{R}$ simultaneously, but we have now found such a set, $K^*$, by Theorem 3.7. Clearly, every substate $s \geq s_0$ that satisfies the known constraints on our world is a member of $K^*$, and every substate $s$ in $K^*$ satisfies all the known constraints. This means that the concept of entailment in Definition 2.1 can be reinterpreted, for a set of rules including negations and embedded implications, to read as follows: A ground instance $A$ is entailed by $s_0$ and $\mathcal{R}$ if and only if $A \in s$ for every $s \in K^*$. Definitions 2.2 and 2.3 can be reinterpreted in the same way: The formula $(\exists x)P(x)$ is entailed by $s_0$ and $\mathcal{R}$ if and only if, for every $s \in K^*$, there exists a ground substitution $\theta$ such that $P(x)\theta \in s$. The formula $(\exists x)P(x)$ is uniformly entailed by $s_0$ and $\mathcal{R}$ if and only if there exists a ground substitution $\theta$ such that, for every $s \in K^*$, $P(x)\theta \in s$. Alternatively, we can interpret Theorem 3.7 in terms of Kripke models: It is easy to see that $K^*$ is the unique maximal Kripke model for the set of rules $\mathcal{R}$ among all the sets $J \subseteq B$. The set $K^*$ thus plays a role in the theory of clausal intuitionistic logic that is similar to the role of the set $G$ in the theory of Horn-clause logic.

Despite this close analogy to the fixed-point semantics for Horn-clause logic, there is an important difference. Let us consider, by analogy to Lemma 2.11 and Theorem 2.12, the successive applications of $T(J) = T_N(J) \cap J_0$ to the top element of the lattice $[\mathcal{P}(B), \subseteq]$. We first prove the following:

**Lemma 3.8.** $\bigcap \{ T^k(B) | k < \omega \} = \bigcap \{ T^k_N(J_0) | k < \omega \}$.

**Proof.** We note several facts about the transformation $T_N$. First, by Definition 3.5, $T_N(J) \subseteq J$ for any set $J$. Second, by monotonicity, $T_N(\bigcap J) \subseteq \bigcap \{ T_N(J) \}$ for any finite collection of sets $\{ J \}$. Finally, since $B \supseteq J_0$, $T_N(B) \supseteq T_N(J_0)$ by monotonicity, and thus $T^k_N(B) \supseteq T^k_N(J_0)$ for all $k$ by induction.
These facts are sufficient for the proof of the lemma. We first establish the inequality in the lemma from right to left, using the following inequalities for $T^k(B)$:

$$T(B) = T_N(B) \cap J_0 \supseteq T_N(J_0) \cap J_0 = T_N(J_0),$$

$$T^2(B) = T(T(B)) \supseteq T(T_N(J_0)) = T^2_N(J_0) \cap J_0 = T^2_N(J_0),$$

$$\vdots$$

$$T^k(B) = T(T^{k-1}(B)) \supseteq T(T^k_N(J_0)) = T^k_N(J_0) \cap J_0 = T^k_N(J_0).$$

Thus $\cap\{T^k(B) : k < \omega\} \supseteq \cap\{T^k_N(J_0) : k < \omega\}$.

We now establish the inequality in the lemma from left to right, using the following inequalities for $T^k(B)$:

$$T(B) = T_N(B) \cap J_0,$$

$$T^2(B) = T_N(T_N(B) \cap J_0) \cap J_0 \subseteq T^2_N(N(B) \cap J_0),$$

$$\vdots$$

$$T^k(B) \subseteq T^k_N(B) \cap T^k_N(J_0) \cap \cdots \cap J_0.$$

Thus

$$\cap\{T^k(B) : k < \omega\} \subseteq \cap\{T^k_N(B) \cap T^k_N(J_0) \cap \cdots J_0 : k < \omega\}$$

$$= \cap\{T^k_N(B) \cap T^k_N(J_0) \cap \cdots \cap J_0 : k < \omega\}$$

$$= \cap\{T^k_N(J_0) : k < \omega\},$$

and the proof of the lemma is complete. □

For convenience, we now define $J_\infty = \cap\{T^k(B) : k < \omega\} = \cap\{T^k_N(J_0) : k < \omega\}$. Then, by analogy to Theorem 2.12, we would like to show that $K^* = J_\infty$.

Unfortunately, this is not the case. To see this, let us extend the iterations of $T^k(B)$ through all ordinals $\alpha$. If $\alpha = \beta + 1$ is a successor ordinal, we will define $T^\alpha(B) = T(T^\beta(B))$, and if $\alpha$ is a limit ordinal, we will define $T^\alpha(B) = \cap\{T^\beta(B) : \beta < \alpha\}$. The following result is well known:

**Theorem 3.9.** Let $K^*$ be the greatest fixed point of the monotonic transformation $T$ on the complete lattice $[\mathcal{P}(B), \subseteq]$. Then there exists an ordinal $\alpha$ of cardinality less than or equal to the cardinality of $\mathcal{P}(B)$ such that $K^* = T^\alpha(B)$.


However, unless the transformation $T$ is continuous, we have no assurance that this result is true for $T^\alpha(B) = \cap\{T^k(B) : k < \omega\}$. Consider the analogue of Lemma 2.11, with join continuity replaced by meet continuity. Although we can easily show that

$$T(\cap\{I_k : k < \omega\}) \subseteq \cap\{T(I_k) : k < \omega\}$$
for any descending chain \( I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \) in \( \mathcal{P}(B) \), the proof of the opposite inequality in Lemma 2.11 will fail when applied to the operator \( T(J) = T_N(J) \cap J_0 \). This means that the first half of the proof of Theorem 2.12 will also fail, and we will only be able to prove the following:

**Lemma 3.10.** \( K^* \subseteq \bigcap \{ T^k(B) \mid k < \omega \} = \bigcap \{ T^k_\mathcal{N}(J_0) \mid k < \omega \} \).

**Proof.** \( K^* \subseteq B \). Thus \( K^* = T(K^*) \subseteq T(B) \), by the fixed-point property and the monotonicity of \( T \). Thus, by induction, \( K^* \subseteq T^k(B) \) for all \( k \). \( \square \)

The inequality in the opposite direction will not, in general, be true. Indeed, in Example 3.19 below, we will see that \( K^* = T^{\omega+1}(B) \), but \( K^* \) is only a proper subset of \( T^\omega(B) \). The situation is thus reminiscent of the situation analyzed in [1], except that Apt and van Emden were there discussing the transformation \( T(s) = s_0 \sqcup T'(s) \) applied to the top element of the lattice \([B, \leq] \), instead of the transformation \( T(J) = T_N(J) \cap J_0 \) applied to the top element of the lattice \([\mathcal{P}(B), \subseteq] \).

### 3.3. The Model Intersection Property

Although we have just shown that our analogy to the fixed-point semantics for Horn-clause logic fails in one important respect, there is another important component of the analogy, the model intersection property, which is valid. Recall that this property holds for the set \( G \) consisting of all the substates \( s \geq s_0 \) that satisfy the rules \( P \iff \bigwedge Q_j \) and \( P \iff \bigvee Q_j \). In our analysis of the negation and embedded implication rules, we have constructed various subsets of \( G \), and we will now show that the model intersection property holds for all of these subsets as well. We will do this in a series of lemmas, culminating in a proof that the model intersection property holds for the set \( K^* \). In all of these lemmas, we will assume that the relevant sets are nonempty. This assumption was not necessary in the Horn-clause case, since the set \( G \) always contains the top element of the lattice \([B, \leq] \).

**Lemma 3.11.** Assume \( J_0 \) is nonempty. Then for any nonempty set \( J \),

\[ J \subseteq J_0 \Rightarrow \neg J \subseteq J_0. \]

**Proof.** Consider the set \( \{ \neg J \} \cup J \). We will show that this set is a subset of \( G \) and that it satisfies the rules \( P \Rightarrow \neg Q \). We know from Lemma 2.7 that \( \neg J \in G \), and thus \( \{ \neg J \} \cup J \subseteq G \). We also know that every \( s \in J \subseteq J_0 \) satisfies the rules \( P \Rightarrow \neg Q \), by the definition of \( J_0 \), and so it suffices to show that \( \neg J \) satisfies these rules as well. Assume then that \( P(x) \theta \in \neg J \) for some selected negation rule \( P(x) \Rightarrow \neg Q(x; y) \) and for some ground substitution \( \theta \). Since \( \neg J \) is a lower bound of \( J \), \( P(x) \theta \in s \) for every \( s \in J \), and since \( J \) satisfies the negation rule \( P(x) \Rightarrow \neg Q(x; y) \), it follows that \( Q(x; y) \theta' \notin s' \) for every \( s' \in J \) and for every \( \theta' \geq \theta \). Since \( J \) is nonempty, this means that \( Q(x; y) \theta' \notin \neg J \) for every \( \theta' \geq \theta \), and we conclude that the set \( \{ \neg J \} \cup J \) satisfies the negation rule \( P \Rightarrow \neg Q(x; y) \). Since the selected rule \( P \Rightarrow \neg Q \) was arbitrary, it follows that \( \{ \neg J \} \cup J \) satisfies all rules of this form.

We have thus shown that \( \{ \neg J \} \cup J \) is a subset of \( G \) satisfying the negation rules \( P \Rightarrow \neg Q \). Since \( J_0 \) is, by definition, the union of all sets for which this is true, it follows that \( \neg J \in J_0 \). \( \square \)
Lemma 3.12. Assume $T_N^k(J_0)$ is nonempty. Then for any nonempty set $J$,

$$J \subseteq T_N^k(J_0) \implies \forall J \in T_N^k(J_0)$$

PROOF. The proof is by induction on the exponent $k$. The case $k = 0$ is given by Lemma 3.11. Let us then assume that the proposition is true for $T_N^k(J_0)$ and consider any nonempty $J \subseteq T_N^{k+1}(J_0)$. From the definition of $T_N$, we know that $J \subseteq T_N^k(J_0)$, and from our induction assumption we know that $\forall J \in T_N^k(J_0)$. We therefore need to show that $\forall J$ is preserved by the application of $T_N$ to $T_N^k(J_0)$.

Assume first that there exists a negation rule $P \leftarrow Q$ and a ground substitution $\theta$ restricted to $U(\forall J)$ such that $Q(x; y)\theta \notin s'$ for all $s' \geq \forall J$ in $T_N^k(J_0)$ and for all $\theta' \geq \theta$. Consider this same negation rule applied to an arbitrary state $s \in J$. It is clear that $Q(x; y)\theta' \notin s'$ for all $s' \geq s$ in $T_N^k(J_0)$ and for all $\theta' \geq \theta$, and thus $s$ will be preserved by the application of $T_N$ to $T_N^k(J_0)$ only if $P(x)\theta \in s$. But we already know that $s$ is preserved by the application of $T_N$ to $T_N^k(J_0)$, because $s \in J \subseteq T_N^k(J_0)$. From this analysis, then, we can conclude that $P(x)\theta \in s$ for all $s \in J$ and thus $P(x)\theta \in \forall J$. For the embedded implication rules $P \Leftarrow [Q \Rightarrow R]$ the analysis is identical. Since these results hold for all the rules involved in Definition 3.5, it is clear that $\forall J$ is itself preserved by the application of $T_N$ to $T_N^k(J_0)$. In other words, $\forall J \in T_N^{k+1}(J_0)$. $\Box$

Lemma 3.13. Assume $J_\infty = \bigcap\{T_N^k(J_0) \mid k < \omega\}$ is nonempty. Then for any nonempty set $J$,

$$J \subseteq J_\infty \implies \forall J \in J_\infty$$

PROOF. Assume $J \subseteq J_\infty$. Then for every $k$, $J \subseteq T_N^k(J_0)$ and thus $\forall J \in T_N^k(J_0)$ by Lemma 3.12. But this means that $\forall J \in J_\infty$. $\Box$

Lemma 3.14. Assume $K^*$ is nonempty. Then for any nonempty set $J$,

$$J \subseteq K^* \implies \forall J \in K^*$$

PROOF. The proof is by transfinite induction. By Theorem 3.9, there exists some ordinal $\alpha$ of cardinality less than or equal to the cardinality of $P(B)$ such that $K^* = T_\alpha(B)$. It therefore suffices to prove that the following proposition:

$$J \subseteq T_\alpha(B) \implies \forall J \in T_\alpha(B)$$

is true for all ordinals $\alpha$.

For the case $\alpha = \omega$, the proposition is true by Lemma 3.8 and Lemma 3.13, and we can use this fact as the basis of the induction. Assume, then, that the proposition is true for all transfinite ordinals $\beta < \alpha$. If $\alpha$ is a limit ordinal, then the proposition can be proven true for $\alpha$ by the same argument used in Lemma 3.13. Specifically, assume that

$$J \subseteq T_\alpha(B) = \bigcap\{T_\beta(B) \mid \beta < \alpha\}.$$ 

Then $\forall J \in T_\beta(B)$ for all $\beta$ by the induction hypothesis, and it follows that $\forall J \in T_\alpha(B)$. Alternatively, if $\alpha = \beta + 1$ is a successor ordinal, assume that

$$J \subseteq T_\alpha(B) = T(T_\beta(B)) = T_N(T_\beta(B)) \cap J_0,$$

and apply the argument used in Lemma 3.12. Since $J \subseteq T_\beta(B)$, we know from the
induction hypothesis that $\bigcap J \in T^\beta(B)$, and we need to show that $\bigcap J$ is preserved by the application of $T_N$ to $T^\beta(B)$. But this fact follows exactly as in the proof of Lemma 3.12. Thus $\bigcap J \in T_N(T^\beta(B))$. Also, since $J \subseteq J_0$, we know from Lemma 3.11 that $\bigcap J \in J_0$. Thus $\bigcap J \in T^\gamma(B)$, and the proof of the lemma is complete. $\Box$

Note the assumption in Lemma 3.14 that $K^*$ is nonempty. If $K^*$ were empty, of course, there would be no substates in $B$ that satisfy $s_0$ and $\mathcal{R}$. Thus the statement that $K^*$ is nonempty is a way of saying, semantically, that the initial substate $s_0$ and the rules $\mathcal{R}$ are mutually consistent.

The following theorem now follows as a trivial consequence of the preceding lemmas:

**Theorem 3.15.** Let $T^k_N(J_0)$ denote the $k$th successive application of $T_N$ to $J_0$, and let

$J_\infty = \bigcap\{ T^k_N(J_0) \mid k < \omega \}$. Let $K^*$ be the greatest fixed point of the transformation $T(J) = T_N(J) \cap J_0$. If $K^*$ is nonempty, then

$\bigcap T^k_N(J_0) \subseteq T^k_N(J_0)$ for all $k < \omega$,

$\bigcap J_\infty \in J_\infty$,

$\bigcap K^* \in K^*$.

Furthermore,

$\bigcap K^* \geq \bigcap J_\infty \supseteq \bigcup\{ \bigcap T^k_N(J_0) \mid k < \omega \}$.

**Proof.** Obvious, from Lemma 3.10 and Lemmas 3.11 through 3.14. $\Box$

The situation is illustrated in Figure 3. The most important part of this theorem is the claim that $\bigcap K^* \subseteq K^*$. We have shown in Theorem 3.7 that $K^*$ is the largest subset of $B$ that satisfies the rules in $\mathcal{R}$, and we now see that $K^*$ itself contains a unique minimal substate $\bigcap K^*$. As a result, we see that entailment according to Definition 2.2 implies uniform entailment according to

![Figure 3](image-url)
Definition 2.3, exactly as in the Horn-clause case:

Corollary 3.16. Let \( s_0 \) be a (possibly infinite) initial state in \( B \), and let \( \mathcal{R} \) be a (possibly infinite) set of rules including negations and embedded implications. If \( (\exists x)P(x) \) is entailed by \( s_0 \) and \( \mathcal{R} \), then \( (\exists x)P(x) \) is uniformly entailed by \( s_0 \) and \( \mathcal{R} \).

**Proof.** The proof is exactly the same as the proof of Corollary 2.10. Assume that \( (\exists x)P(x) \) is entailed by \( s_0 \) and \( \mathcal{R} \). Then, since \( \bigcap K^* \subseteq K^* \), there must exist some ground substitution \( \theta \) such that \( P(x)\theta \in \bigcap K^* \). But this means that \( P(x)\theta \in s \) for every \( s \in K^* \). \( \square \)

This result is not new: It follows directly from the early results of Harrop [10] and Kleene [12], since the rules \( \mathcal{R} \) satisfy Harrop’s condition insuring that a closed formula contains no positive occurrences of disjunctions or existential quantifiers. However, the theorems of Harrop and Kleene were established by proof-theoretic techniques, using a Gentzen-style proof system for intuitionistic logic, and our derivation of this result from a fixed-point construction in the semantics appears to be novel.

The importance of this result lies in its similarity to Corollary 2.10 for Horn-clause logic. As a practical matter, Corollary 3.16 shows that any successful query ‘\( P(x) \)’ in a set of rules including negations and embedded implications has a definite answer substitution for the variables \( x \). Clearly, this would not be the case if the negation and embedded implication rules were interpreted classically. For example, the rule \( P(x) \leftarrow \neg Q(x; y) \) would be classically equivalent to the formula \( (\forall x)[P(x) \lor (\exists y)Q(x; y)] \), which certainly does not have a unique minimal model, and a query ‘\( (\exists x)P(x) \)’ might be classically entailed by these rules without having a definite answer substitution for the variables \( x \). In this sense, then, clausal intuitionistic logic is the more natural generalization of Horn-clause logic, as demonstrated by Theorem 3.15 and Corollary 2.10.

However, the implications of Theorem 3.15 for the proof of a completeness theorem are not so favorable. Consider the inequality in the last line of this theorem.

Let us suppose, in a particular case, that this inequality is actually an equality. This would then give us a simple way to compute the ground instances in \( \bigcap K^* \), in the same way we computed the ground instances in \( g^* = \bigcap G \). Under this assumption, if \( P(x)\theta \in \bigcap K^* \), then \( P(x)\theta \) would be a member of \( \bigsqcup \{ \bigcap T_{\leq n}(J_0) \mid k \leq n \} \) for some finite \( n \), and we could compute \( P(x)\theta \) explicitly by applying \( T_N \) to \( J_0 \) a finite number of times. We would thus have the foundations of a constructive completeness proof for the negation and embedded implication rules, analogous to the constructive completeness proof for the rules \( P \leftarrow \land Q_j \) and \( P \leftarrow \lor Q_j \) in Theorem 2.14. Unfortunately, the inequality in Theorem 3.15 is, in general, just an inequality, for essentially the same reasons that the transformation \( T(J) = T_N(J) \cap J_0 \) is, in general, noncontinuous. The counterexamples are discussed in Section 3.4 below. This means that the proof of a completeness theorem for a system of rules including negations and embedded implications will be much more complicated than the proof of a completeness theorem in the pure Horn-clause case.

### 3.4. Examples

We will now analyze three examples of the fixed-point semantics for clausal intuitionistic logic. Example 3.17 is well behaved: The sequence \( J_0, T_N(J_0) \).
Theorem 3.15 converges in order \( \omega \) to the fixed point \( K^* \). This example will also be used in Section 4.1 to illustrate the proof procedures for our system. However, as we have pointed out, this convergence property does not hold in all cases. Examples 3.18 and 3.19 are pathological: In these cases, the inequalities in Lemma 3.10 and Theorem 3.15 are strict inequalities. We will discuss the reasons for this behavior at the end of the section.

Example 3.17. We will analyze here the interaction of four negation rules, two of them abstraction rules and two of them expansion rules. Assume that \( \mathcal{R} \) consists of the following:

\[
P_1(x) \iff -Q_1(x, y),
\]

\[
P_2(x) \implies -Q_2(x, y),
\]

\[
Q_1(x, y) \implies -R(x, y, z),
\]

\[
Q_2(x, y) \iff -R(x, y, z),
\]

and assume that the initial substate \( s_0 = \{ P_2(a) \} \). Since there are no Horn clauses in this set, \( G \) simply consists of all substates containing \( 'P_2(a)' \), and \( g^* = \bigcap G = \{ P_2(a) \} \).

To construct the set \( J_0 \), we must analyze the effect of the expansion rules (14) and (15). Because of the rule (14), all substates containing \( 'Q_2(a, y)' \) will be deleted from \( J_0 \), for any instantiation of the variable \( y \). Because of the rule (15), there are certain substates greater than the substates containing \( 'Q_1(a, y)' \) that will also be deleted from \( J_0 \), for any instantiation of the variable \( y \). For example, if \( s \) is a substate containing \( 'Q_1(a, b)' \), then all substates \( s' \geq s \) that contain \( 'R(a, b, z)' \) would be deleted from \( J_0 \), for any instantiation of the variable \( z \). Note at this point that \( \bigcap J_0 \) is still equal to \( \{ P_2(a) \} \).

We now apply \( T_N \) to \( J_0 \), and we focus our attention on any substate \( s \in J_0 \) containing \( 'Q_7(a, b)' \). The relevant rule is (16). We know from our construction of \( J_0 \) that there are no substates \( s' \geq s \) that contain \( 'R(a, b, z)' \), for any instantiation of the variable \( z \), and thus \( T_N \) will delete \( s \) from \( J_0 \) unless \( s \) also contains \( 'Q_4(a, b)' \), by (16). But this is impossible, since there are no substates in \( J_0 \) that contain \( 'Q_4(a, b)' \). Furthermore, since this same argument applies to any substate \( s \in J_0 \) containing \( 'Q_5(a, y)' \), for any instantiation of the variable \( y \), it is clear that all such substates are deleted from \( J_0 \) by the application of \( T_N \). Note, however, that the initial substate \( s_0 \) is preserved by the application of \( T_N \) to \( J_0 \), and thus \( \bigcap T_N(J_0) = \{ P_2(a) \} \).

Finally, we consider the set \( J_1 = T_N(J_0) \), and we apply \( T_N \) once more to \( J_1 \). The relevant rule is now (13). At this point, since there are no substates at all in \( J_1 \) that contain \( 'Q_1(a, y)' \) for any instantiation of the variable \( y \), the initial substate \( s_0 \) is itself deleted by \( T_N \), and all the substates that are preserved by \( T_N \) must contain \( 'P_1(a)' \). We can now verify that the system has reached a fixed point, and that we have obtained the following solution:

\[
\bigcap K^* = \bigcap T_N^2(J_0) = \{ P_2(a), P_1(a) \}.
\]

In this case, of course, the inequality in the last line of Theorem 3.15 is actually an equality.

Example 3.18. We will now analyze two counterexamples to the opposite inequality in Theorem 3.15. We will first construct an example in which \( \bigcap J \) is
strictly greater than $\bigcup \{ \bigcap T^k_N(J_0) \mid k < \omega \}$. Assume that $\mathcal{R}$ consists of the following rules:

\begin{align*}
    P(x) & \iff \neg Q(x, y), \quad (17) \\
    R(x, y) & \Rightarrow \neg Q(x, y), \quad (18) \\
    R(x, f(y)) & \iff \neg Q(x, y), \quad (19)
\end{align*}

and assume that $s_0$ is a specially constructed initial substate, which we define as follows: Let us write $U/f$ for the set of all terms in $U$ except for those terms that begin with the function symbol ‘$f$’. For example, the term ‘$g(f(x), y)$’ would be included in $U/f$, but the terms ‘$f(g(f(x), y))$’, ‘$f(f(g(f(x), y)))$’, and so on, would not be included in $U/f$. We then set

$$s_0 = \{ R(a, a, \text{term}) \mid \text{term} \in U/f \},$$

and we consider successive applications of $T_N$ to $J_0$.

First, consider $J_0$ itself. For every term $\in U/f$, and for every $s \in J_0$, it follows from rule (18) that ‘$Q(a, \text{term})$’ $\notin s$. Thus, when we apply $T_N$ to $J_0$, the rule (19) guarantees that every substate $s \in T_N(J_0)$ will contain ‘$R(a, f(\text{term}))$’ for every term $\in U/f$. The rule (18) then assures us that ‘$Q(a, f(\text{term}))$’ $\in s$ for every $s \in T_N(J_0)$, and so on. In general, by induction, if $s \in T_N(J_0)$, then $s$ contains ‘$R(a, f^k(\text{term}))$’ for every term $\in U/f$. This means that $\bigcup \{ \bigcap T^k_N(J_0) \mid k < \omega \}$ contains ‘$R(a, \text{term})$’, ‘$R(a, f(\text{term}))$’, ‘$R(a, f^2(\text{term}))$’, ‘$R(a, f^3(\text{term}))$’, ..., for all finite sequences of applications of the function symbol ‘$f$’. We now analyze the rule (17). Is the right-hand side of (17) ever satisfied on $\bigcap T^k_N(J_0)$, for any $k$? The answer is negative, since there will always be some ground instance ‘$Q(a, f^{k+1}(\text{term}))$’ remaining in some substate $s$ in $T^k_N(J_0)$. Thus, $\bigcup \{ \bigcap T^k_N(J_0) \mid k < \omega \}$ does not contain the ground instance ‘$P(a)$’.

Consider now an arbitrary substate $s \in J_0$. Since $s \in T^k_N(J_0)$ for all $k$, it follows from our previous analysis that $s$ contains ‘$R(a, f^k(\text{term}))$’ for all $k$. But $s$ is also a member of $J_0$, and thus the rule (18) implies that, for all $k$, ‘$Q(a, f^k(\text{term}))$’ $\notin s'$ for all $s' \geq s$ in $J_0$. Is the ground instance ‘$P(a)$’ contained in $s$? Note that the set

$$\{ f^k(\text{term}) \mid \text{term} \in U/f, k < \omega \}$$

completely exhausts the set of terms in $U$, and thus the right-hand side of (17) would be satisfied on the substate $s$ in $J_0$. If we assume that ‘$P(a)$’ $\notin s$, then $s$ would be deleted by the application of $T_N$ to $J_0$, and this would contradict the assumption that $s \in J_0$. Thus ‘$P(a)$’ $\in J_0$. Since $s$ was chosen as an arbitrary substate in $J_0$, however, it follows that ‘$P(a)$’ $\in \bigcap J_0$. We have thus shown that $\bigcap J_0$ is strictly greater than $\bigcup \{ \bigcap T^k_N(J_0) \mid k < \omega \}$.

Example 3.19. We will now construct an example in which $\bigcap K^*$ is strictly greater than $\bigcap J_0$. Assume that $\mathcal{R}$ consists of the following rules:

\begin{align*}
    P(x) & \iff \neg Q(x, y), \quad (20) \\
    R(x, y) & \Rightarrow \neg Q(x, y), \quad (21) \\
    Q(x, f(y)) & \iff \neg S(x, f(y)), \quad (22) \\
    Q(x, y) & \iff \neg S(x, f(y)), \quad (23)
\end{align*}
and assume that $s_0$ is the same initial substate constructed in Example 3.18:

$$s_0 = \{ 'R(a, \text{term})' \mid \text{term} \in U/f \}.$$

With these assumptions, we will show that $T_N(J_\omega)$ is strictly less than $J_\omega$, so that $J_\omega$ cannot be a fixed point of $T_N$. This is therefore a counterexample to the opposite inequality in Lemma 3.10, as well as a counterexample to the opposite inequality in Theorem 3.15.

Clearly, the initial analysis of $J_0$ is identical to the analysis of $J_\omega$ in Example 3.18. In other words, for every term $\in U/f$ and for every $s \in J_\omega$, it follows from the rule (21) that $'Q(a, \text{term}') \not\in s$. However, these rules allow the existence of a substate $s \in J_0$ for which $'Q(a, f(\text{term}))' \in s$, and we will focus our attention for the moment on such a substate $s$. By the rule (22), since $'Q(a, f(\text{term}))' \in s$, it follows that $'S(a, f(\text{term}))' \not\in s'$ for all $s' \geq s$ in $J_0$. Now apply $T_N$ to $J_\omega$. By the rule (23), $T_N$ will delete the substate $s$ unless $s$ also contains $'Q(a, \text{term})'$, which is impossible by our initial analysis of $J_\omega$. Thus, for every term $\in U/f$, we see that there exists no substate $s \in T_N(J_\omega)$ for which $'Q(a, f(\text{term}))' \in s$. By induction, we can now extend this analysis to an arbitrary $T^k_N(J_\omega)$. For every $k$, there will exist some substate $s \in T^k_N(J_\omega)$ such that $'Q(a, f^{k+1}(\text{term}))' \in s$. but $'Q(a, f^k(\text{term}))' \not\in s$ for every substate $s \in T^k_N(J_\omega)$.

We now claim that $s_0 \in J_\omega$, but $s_0 \not\in T_N(J_\omega)$. To see this, examine any $T^k_N(J_0)$, and note that the right-hand side of (20) cannot be satisfied on $\Gamma \cap T^k_N(J_0)$, since there is always some substate in $T^k_N(J_\omega)$ that contains $'Q(a, f^{k+1}(\text{term}))'$. This means that the initial substate $s_0$, which does not contain $'P(a)'$, will always be preserved by successive applications of $T_N$ to $J_\omega$. In other words, $s_0 \in T^k_N(J_\omega)$ for all $k$, and thus $s_0 \in J_\omega$. On the other hand, for every $k$, there can be no substate in $J_\omega$ containing $'Q(a, f^k(\text{term}))'$, since every such substate is deleted by $T^k_N(J_\omega)$. Thus, when we apply $T_N$ to $J_\omega$, the rule (20) causes the deletion of $s_0$ from $T_N(J_\omega)$. We have thus shown that $J_\omega$ cannot be a fixed point of $T_N$. By Lemma 3.10, this means that $K^*$, which is a fixed point of $T_N$, must be a proper subset of $J_\omega$, and that $\Gamma \cap K^*$ must be strictly greater than $\Gamma \cap J_\omega$.

It should now be obvious why these counterexamples behave as they do. Since all the individual constants in $U$ appear in the initial substate $s_0$, the abstraction and expansion rules working together eventually construct every term in the universe of discourse, and thus the right-hand side of the rule $'P(a) \iff Q(a, y)'$ is eventually satisfied for every possible value of the universally quantified $'y'$ variable. This result would not occur if $U$ contained additional individual constants. In Section 4.5, in fact, we will elevate this observation to the status of a theorem, and show that the inequality in Theorem 3.15 is actually an equality if there exists an infinite supply of individual constants in $U$ that do not appear in $s_0$. To do this, however, we need to develop a proof procedure for our system of clausal intuitionistic logic, and show that it is both sound and complete. This is the subject of the following section.

NOTE. Sections 4, 5, and 6 will appear in the second paper of this pair.
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