Time-optimal path parameterization in SO(3) with applications to the spacecraft reorientation problem

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Introduction

Time-optimal reorientation of a rigid body is an interesting fundamental problem that arises in a number of applications including spacecraft or robot motion planning, computer animation, etc. There are two main directions of research in the literature. The aerospace community has considered the problem of finding *exact* time-optimal reorientation trajectories under kinodynamic constraints by applying directly Pontryagin's minimum principle (Li and Bainum 1990; Bilimoria and Wie 1993; Bai and Junkins 2009). However, this approach could only deal with the relatively simple cases of rest-to-rest motions in an obstacle-free environment.

In the robotics community, popular sampling-based motion planning methods such as RRT or PRM have been extended to SO(3)[1]. Such methods require three components : (i) random sampling, (ii) distance metric, and (iii) interpolation between two points. While straightforward in Euclidean spaces, these three components involve considerable challenges in SO(3). Regarding component (i), a sampling algorithm based on unit quaternions was proposed in (Kuffner 2004). As for components (ii) and (iii), methods based on unit quaternions or on rotation matrices exist, but have rarely been considered in the contexts of kinodynamic constraints and time-optimality.

As a result, to our knowledge, there are no general methods for planning fast trajectories in SO(3) under kinodynamic constraints and in a cluttered environment. Here we address this problem by following the widely used plan-and-shortcut pipeline, which has shown to be easy to implement, robust and efficient (it is e.g. the default pipeline used in OpenRAVE, a popular robotics software in both academia and industry). This pipeline consists of three steps

1. plan piece-wise "linear" *paths* considering collision constraints using RRT;
2. reparameterize these paths considering time-optimality and kinodynamic constraints;
3. shortcut using time-optimal interpolants under kinodynamic constraints.

[1]The set of all 3×3 orthogonal matrices with determinant +1 forms a group known as the special orthogonal group SO(3), also known as group of rotation matrices.

Step 1 is the basic RRT in SO(3) (Kuffner 2004). Note that "linear" paths in Euclidean spaces correspond to great-circle arcs in SO(3). Step 2 and step 3 are based on our extension of Time-Optimal Path Parameterization (TOPP, cf. (Pham 2014)) to SO(3) and constitute the core contribution of this paper.

TOPP in SO(3)

Consider a path $\mathcal{P}$ – represented as the underlying path of a trajectory $\boldsymbol{r}(s)_{s\in[0,s_{\mathrm{end}}]}$ – in the configuration space. Assume that $\boldsymbol{r}(s)_{s\in[0,s_{\mathrm{end}}]}$ is C^1- and piecewise C^2-continuous. We are interested in *time-parameterizations* of $\mathcal{P}$, which are increasing *scalar functions* $s : [0,T] \rightarrow [0, s_{\mathrm{end}}]$, under kinodynamic constraints.

If the constraints can be expressed in the form (note that all vector inequalities in this paper are element-wise)

$$\ddot{s}\boldsymbol{a}(s) + \dot{s}^2\boldsymbol{b}(s) + \boldsymbol{c}(s) \leq \boldsymbol{0}, \tag{1}$$

then there exists efficient methods and implementations to find the time-optimal parameterization $s(t)$ (see (Pham 2014) for references and details).

Consider the case of a rigid body with three independent actuations, such as rigid spacecraft, whose equations of motion are $\mathbb{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbb{I}\boldsymbol{\omega}) = \boldsymbol{\tau}$ where $\mathbb{I}$ is the 3×3 inertia matrix of the spacecraft and $\boldsymbol{\tau}$ the 3-dimensional torque vectors. The actuation bounds are given by $\boldsymbol{\tau}_{\min} \leq \boldsymbol{\tau} \leq \boldsymbol{\tau}_{\max}$.

Following (Park and Ravani 1995; Park and Ravani 1997), consider a path in the space of rotation matrices $\mathbf{R}(s)_{s\in[0,1]} \in \mathrm{SO}(3)$ given by

$$\mathbf{R}(s) = \mathbf{R}_0 \boldsymbol{e}^{[\boldsymbol{r}(s)]},$$

where $\boldsymbol{r}(s)$ is a 3-dimensional vector. The interest of this types of paths is that they are smooth (if $\boldsymbol{r}$ is smooth), are bi-invariant and interpolate optimally between two rotations (in the case of rest-to-rest motions) and near-optimally between two rotations when initial and final angular velocities are specified. For a detailed discussion, see (Park and Ravani 1997).

Following again (Park and Ravani 1997), we can next write

$$\begin{aligned} \boldsymbol{\tau} &= \ddot{s}\mathbb{I}\mathbf{A}(\boldsymbol{r})\boldsymbol{r}_s \\ &+ \dot{s}^2 \left\{ \mathbb{I}\mathbf{A}(\boldsymbol{r})\boldsymbol{r}_{ss} + \mathbb{I}\mathbf{C}(\boldsymbol{r}, \boldsymbol{r}_s) + (\mathbf{A}(\boldsymbol{r})\boldsymbol{r}_s) \times (\mathbb{I}\mathbf{A}(\boldsymbol{r})\boldsymbol{r}_s) \right\}. \end{aligned}$$

where $\mathbf{A}(\boldsymbol{r}) \stackrel{\text{def}}{=} \mathbf{I} - \frac{1-\cos\|\boldsymbol{r}\|}{\|\boldsymbol{r}\|^2}[\boldsymbol{r}] + \frac{\|\boldsymbol{r}\|-\sin\|\boldsymbol{r}\|}{\|\boldsymbol{r}\|^3}[\boldsymbol{r}]^2$, and

$$\begin{aligned}\mathbf{C}(\boldsymbol{r},\dot{\boldsymbol{r}}) \stackrel{\text{def}}{=} & \frac{\|\boldsymbol{r}\|-\sin\|\boldsymbol{r}\|}{\|\boldsymbol{r}\|^3}\dot{\boldsymbol{r}}\times(\boldsymbol{r}\times\dot{\boldsymbol{r}}) \\ - & \frac{2\cos\|\boldsymbol{r}\|+\|\boldsymbol{r}\|\sin\|\boldsymbol{r}\|-2}{\|\boldsymbol{r}\|^4}\boldsymbol{r}^\top\dot{\boldsymbol{r}}(\boldsymbol{r}\times\dot{\boldsymbol{r}}) \\ + & \frac{3\sin\|\boldsymbol{r}\|-\|\boldsymbol{r}\|\cos\|\boldsymbol{r}\|-2\|\boldsymbol{r}\|}{\|\boldsymbol{r}\|^5}\boldsymbol{r}^\top\dot{\boldsymbol{r}}(\boldsymbol{r}\times(\boldsymbol{r}\times\dot{\boldsymbol{r}})).\end{aligned}$$

Thus, the torque bounds can be put in the form of (1)

Planning fast, collision-free trajectories in SO(3) under kinodynamic constraints

In the first step, a planner RRT is used to find a collision-free piece-wise "linear" path. To avoid discontinuities in the velocity vector at each junctions of the linear segment, one must ensure that the velocities at the beginning and the end of each segment are zero.

After obtaining rotation paths, step 2 and 3 of the pipeline mentioned in the Introduction can be easily implemented using TOPP in SO(3). In step 3, at each shortcut iteration, we select two random time instants t_0, t_1 along the trajectory, which correspond to two orientation matrices $\mathbf{R}_{t_0}, \mathbf{R}_{t_1} \in$ SO(3) and two angular velocity vectors $\boldsymbol{\omega}_{t_0}, \boldsymbol{\omega}_{t_1} \in \mathbb{R}^3$. Using (Park and Ravani 1997), we find the interpolation path $\mathbf{R}^*(t)_{t\in[0,T]}$, where $T = t_1 - t_0$, such that

$$\mathbf{R}^*(0) = \mathbf{R}_{t_0},\ \mathbf{R}^*(T) = \mathbf{R}_{t_1},\ \boldsymbol{\omega}^*(0) = \boldsymbol{\omega}_{t_0},\ \boldsymbol{\omega}^*(T) = \boldsymbol{\omega}_{t_1}.$$

If this path involves any collision, we discard it, otherwise, we use TOPP to optimally retime it under kinodynamic constraints. If the resulting duration is smaller than T then we replace the original trajectory portion by the shortcut.

We tested the above algorithm on a reorientation problem for the Messenger spacecraft subject to bounds on angular velocities and torques (model downloaded from `http://nasa3d.arc.nasa.gov/detail/eoss-messenger`), see Fig. 1. We ran 100 trials, with 200 shortcut iterations per trial. The average running time was 1.59±2.66 s, 1.47±1.58 s, and 7.06±6.42 s respectively for the three steps of the pipeline (Intel i7 3.40 GHz with 4 GB RAM). The average number of successful shortcuts per trial was 11.12 ± 6.31. Fig. 1 shows one typical trajectory found by the algorithm. Fig. 2 shows the corresponding angular velocities and torques as functions of time. One can see that, in agreement with time-optimality, at least one constraint is saturated at any time instant.

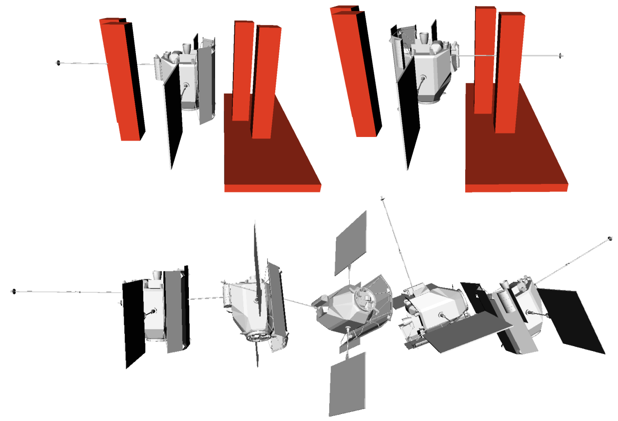

Figure 1: Fast reorientation trajectory for the Messenger spacecraft in a cluttered environment.

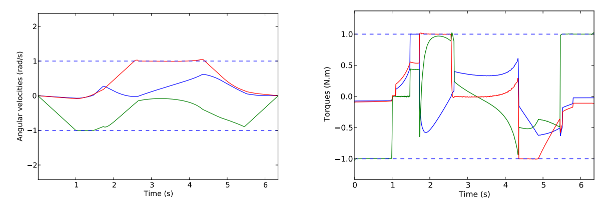

Figure 2: Angular velocities and torques of the trajectory in Fig. 1.

Conclusion

We developed an extension of Time-Optimal Path Parameterization (TOPP) to SO(3). Based on that extension, we proposed an algorithm to plan fast trajectories in SO(3) under kinodynamic constraints and in collision constraints. As illustration, we used the algorithm to plan a fast collision-free reorientation trajectory for a spacecraft subject to velocity and torque bounds, in a cluttered environment. For future work, the algorithms presented here can be extended to SE(3) without any particular difficulties.

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