

# Asymptotically Near Optimal Planning with Probabilistic Roadmap Spanners

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**Abstract**—Asymptotically optimal motion planners guarantee that solutions approach optimal as more iterations are performed. A recently proposed roadmap-based method, the  $\text{PRM}^*$  approach, provides this desirable property and minimizes the computational cost of generating the roadmap. Even for this method, however, the roadmap can be slow to construct and quickly grows too large for storage or fast online query resolution, especially for relatively high-dimensional instances. In graph theory there are algorithms that produce sparse subgraphs, known as graph spanners, which guarantee near-optimal paths. This work proposes different alternatives for interleaving graph spanners with the asymptotically optimal  $\text{PRM}^*$  algorithm. The first alternative follows a sequential approach, where a graph spanner algorithm is applied on the output roadmap of  $\text{PRM}^*$ . The second one is an incremental method, where certain edges are not considered during the construction of the roadmap as they are not necessary for a roadmap spanner. The result in both cases is an asymptotically *near-optimal* motion planning solution. Theoretical analysis and experiments performed on typical, geometric motion planning instances show that large reductions in construction time, roadmap density, and online query resolution time can be achieved with a small sacrifice of path quality through roadmap spanners.

**Index Terms**—Motion planning, near-optimality, probabilistic roadmaps.

## I. INTRODUCTION

Sampling-based algorithms correspond to a practical framework for solving high-dimensional motion planning problems. The Probabilistic Roadmap Method (PRM) [1] is an instance of such a method that uses an offline phase to construct a graph that represents the structure of the configuration space ( $C$ -space). This graph, known as a roadmap, can be queried in an online phase to quickly provide solutions.

Traditionally, PRM and many related variants [2]–[7] focused on quickly finding a feasible solution. Pursuing only this objective may result in reduced solution quality, which can be evaluated given a variety of measures depending on the underlying task. For instance, clearance from obstacles or smoothness can be used to evaluate the quality of the solution path [8]. This work focuses on popular path quality measures that are metric functions, such as path length or traversal time. One standard way to improve solution quality, is through path smoothing, a post-processing step applied on the path returned by a planner. A more sophisticated way to improve path quality is through hybridization graphs, which combine multiple solutions into a higher quality one that uses the best

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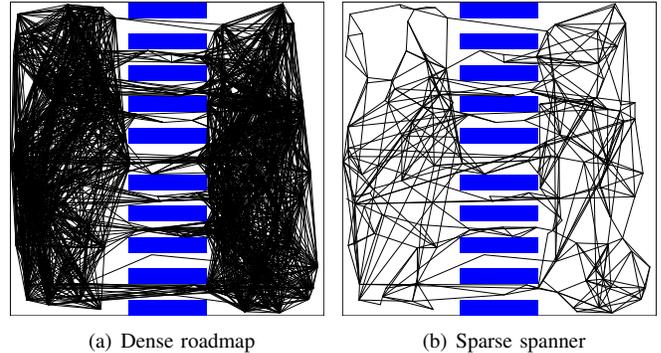


Fig. 1. (a) A roadmap constructed by an algorithm that is guaranteed to converge to optimal solution costs and (b) another roadmap constructed in such a way that guarantees convergence to *near-optimal* solution costs but results in many fewer edges.

portions of each input path [9]. There are also algorithms that attempt to produce roadmaps, which return paths that are deformable to optimal ones after the application of smoothing [10], [11]. These smoothing-based techniques, however, can be expensive for the online resolution of a query, especially when multiple queries must be answered given the same input roadmap.

Alternatively, it is possible to construct larger, denser roadmaps that better sample the  $C$ -space by investing more preprocessing time. For instance, a planner that attempts to connect a new sample to every existing node in the roadmap will converge to solutions with optimal costs, a property known as asymptotic optimality. While roadmaps with this property are desirable for their high path quality, their large size can be problematic. Large roadmaps impose significant costs during construction, storage, transmission and online query resolution, so they may not be appropriate for many applications. The recently proposed  $k$ -PRM\* algorithm [12] minimizes the number  $k$  of neighbors each new roadmap sample has to be tested for connection while still providing asymptotic optimality guarantees. Even so, the density of roadmaps produced by  $k$ -PRM\* can still be high, resulting in slow online query resolution times.

This paper argues that a viable alternative is to construct roadmaps with asymptotic near-optimality guarantees. By relaxing the optimality requirement, it is possible to construct roadmaps that are sparser, faster to build, and can answer queries faster, while providing solution paths that are arbitrarily close to optimal.

The theoretic foundations for this work lie in graph theory. In particular, graph spanners are sparse subgraphs with guarantees on path quality. Spanner construction methods filter

edges of the original graph and guarantee that any shortest path between nodes in the sparse subgraph has cost at most  $t \times c^*$ , where  $c^*$  is the optimum path cost in the original graph and  $t$  is an input parameter called the stretch factor. An existing method in the motion planning literature, which created roadmaps that contained Useful Cycles [13] was in fact producing graph spanners. In this framework, roadmap edges that pass the “usefulness” test are added to the roadmap because not doing so would violate path quality guarantees, similar to those of graph spanners. Nevertheless, since this method did not start from a roadmap with optimality guarantees, it did not provide any properties related to optimality.

In this work, graph spanner algorithms are combined with asymptotically optimal roadmap planners to produce planners that construct roadmap spanners. These are roadmap methods with asymptotic near-optimality guarantees. Two algorithms are proposed in this work:

- The first method builds a roadmap spanner in a sequential manner given an asymptotically optimal roadmap.
- A second proposed alternative interleaves roadmap construction with a spanner preserving filter to directly construct a roadmap spanner in an efficient way.

The resulting roadmaps are sparse, which is beneficial in any application where the density of the roadmap matters.

#### A. Related Work

*Computational efficiency:* There has been a plethora of techniques on how to sample and connect configurations so as to achieve computational efficiency in roadmap construction via a sampling-based process [2]–[6], [14]. Certain algorithms, such as the Visibility-based Roadmap [4], the Incremental Map Generation algorithm [6] and the Reachability Roadmap Method [15] focus on returning a connected roadmap that covers the entire configuration space. A reachability analysis of roadmap techniques suggested that connecting roadmaps is more difficult than covering the  $C$ -space [7]. Furthermore, a method is available to characterize the contribution of a sample to the exploration of the  $C$ -space [16], but it does not address how the connectivity between samples contributes to the resulting path quality once the space has been explored.

*Path Quality:* Work on creating roadmaps that contain high quality paths has been motivated by the objective to efficiently resolve queries without the need for a post-processing optimization step of the returned path. One technique aims to compute all different homotopic solutions [11], while Path Deformation Roadmaps compute paths that are deformable to optimal ones [10]. Another approach inspired by Dijkstra’s algorithm extracts optimal paths from roadmaps for specific queries [17] but may require very dense roadmaps. The Useful Cycles approach implicitly creates a roadmap spanner with small number of edges [13] and has been combined with the Reachability Roadmap Method to construct connected roadmaps that cover 2D and 3D  $C$ -spaces to provide high quality paths [18]. A method has been introduced that filters nodes from a roadmap if they do not improve path quality measures [19].

*Tree-based planners and dynamics:* Roadmaps cannot be constructed for problems where there is no solution for the two-point boundary value problem, i.e., it is not easy to compute a path that connects exactly two states of a moving system. On the other hand, tree-based kinodynamic planners [20], [21] can operate on such environments. Tree-based algorithms are very effective for single-query planning. They do not provide, however, the preprocessing properties of roadmaps and they tend to be slower for multi-query applications. Regarding sparsity, a tree is already a sparse graph and it becomes disconnected when removing edges. It has been shown that the RRT algorithm produces arbitrarily bad paths with high probability and will miss high quality paths even if they are easy to find [12], [22]. Anytime RRT [23] has been proposed as an approach that practically improves path quality in an incremental manner. Tree-based planners for kinodynamic problems can still benefit from an approximate roadmap to estimate distances between states and the goal region that take into account  $C$ -space obstacles. Such distance estimates can be used as a heuristic in tree expansion to bias the selection of states closer to the goal and solve problems with dynamics faster [24], [25].

*Asymptotic Optimality:* The RRG, RRT\*, and PRM\* family of algorithms [12] provide asymptotic optimality for general configuration spaces. RRG and RRT\* are based on RRT, a tree-based planner. The Anytime RRT\* approach [26] extends RRT\* with anytime planning in dynamic environments and can incrementally improve path quality. PRM\* is a modification of standard PRM. The proposed techniques in this work are based on  $k$ -PRM\*, a variation of PRM\*, which will be described in detail in Section II-A.

#### B. Contribution

This paper describes two methods for constructing roadmap spanners based on previous work by the authors [27], [28]. It is shown that both methods guarantee a constant factor of asymptotic optimality, i.e., provide asymptotic near-optimality. The first alternative is the *Sequential Roadmap Spanner* (SRS) method [27], which has the following characteristics:

- It constructs a sparse graph given a dense roadmap.
- It provides asymptotic guarantees on the number of edges.
- It provides asymptotically near-optimal solutions.
- It has the same asymptotic time complexity as PRM.

SRS removes edges from a roadmap with path quality guarantees while retaining near-optimal path quality utilizing state-of-the-art graph spanner algorithms with good complexity performance. In this manner, online query resolution time is shortened while simultaneously reducing the space required for storing and transmitting the roadmap.

The second alternative is the *Incremental Roadmap Spanner* (IRS) method [28], which has the following characteristics:

- It directly constructs sparse roadmaps using a sampling and filtering process.
- In environments where collision detection is expensive, it constructs roadmaps in a computationally efficient way.
- It provides asymptotically near-optimal solutions.
- It has an asymptotic time complexity close to that of PRM.

IRS can incrementally construct a roadmap spanner in a continuous space, while most graph spanner algorithms are formulated to operate on an existing roadmap. Because IRS employs the framework of an asymptotically optimal planner, i.e., PRM\* [12], it provides asymptotic near-optimality guarantees that related approaches, such as the Useful Cycles method, do not [13]. IRS is shown to produce significantly sparser roadmaps than  $k$ -PRM\* experimentally.

Both methods balance two extremes in terms of motion planning solutions. On one hand, the connected component heuristic for PRM can connect the space very quickly with a very sparse roadmap, but can produce solutions with arbitrarily high cost. On the other hand, the  $k$ -PRM\* algorithm provides asymptotically optimal roadmaps that tend to be very dense and slow to construct. With IRS it is possible to tune the solution quality degradation relative to  $k$ -PRM\* and select a parameter, the stretch factor, that will return solutions arbitrarily close to the optimal ones, while still constructing sparse roadmaps efficiently.

The theoretical guarantees on path quality that these techniques provide are tested empirically in a variety of motion planning problems in  $SE(2)$  and  $SE(3)$ . Experiments show that as the desired stretch factor increases, most of the roadmap edges can be removed while increasing mean path length by a relatively small amount. Often, this increase in path length can be significantly reduced by utilizing smoothing. Path degradation is most pronounced for paths that are very short, while longer paths are less affected. The sparsity of the roadmaps produced is a valuable feature in itself, but with IRS, a marked decrease in construction time is also measured.

Relative to the authors' previous work [27], [28], this paper provides a more detailed description and analysis of the two proposed algorithms under a common framework for the development of roadmap spanners. Furthermore, it provides a direct comparison of the two methods using a new, common set of experiments. The experiments suggest that the incremental approach is able to produce sparser roadmaps than the sequential alternative for the same stretch factor.

## II. FOUNDATIONS

A robot can be abstracted as a point in a  $d$ -dimensional configuration-space ( $C$ -space) where the set of collision-free configurations define  $C_{\text{free}} \subset C$  [29]. The experiments performed for this paper take place in the spaces of 2D and 3D rigid body transformations ( $SE(2)$  and  $SE(3)$  correspondingly). The proposed methods are also applicable to any  $C$ -space that is a metric and probability space for reasons described in Section III. Once  $C_{\text{free}}$  can be calculated for a particular robot, one needs to specify initial and goal configurations to define an instance of the path planning problem:

*Definition 1 (The Path Planning Problem):* Given a set of collision-free configurations  $C_{\text{free}} \subset C$ , initial and goal configurations  $q_{\text{init}}, q_{\text{goal}} \in C_{\text{free}}$ , find a continuous curve  $\sigma \in \Sigma = \{\rho | \rho : [0, 1] \rightarrow C_{\text{free}}\}$  where  $\sigma(0) = q_{\text{init}}$  and  $\sigma(1) = q_{\text{goal}}$ .

The PRM algorithm [1] can find solutions to the Path Planning Problem by sampling configurations in  $C_{\text{free}}$  then

trying to connect them to neighboring configurations with local paths. It starts with an empty roadmap and then iterates until some stopping criterion is satisfied. For each iteration, a configuration in  $C_{\text{free}}$  is sampled and a set of neighbors is identified. For each of these neighbors, an attempt is made to connect them to the sampled configuration with a local planner. If such a curve exists in  $C_{\text{free}}$ , i.e., it is collision-free, an edge between the two configurations is added to the roadmap.

The offline phase of the algorithm is completed once some user defined stopping criterion is met. The result is a graph  $G = (V, E)$  that reflects the connectivity of  $C_{\text{free}}$  and can be used to answer query pairs of the form  $(q_{\text{init}}, q_{\text{goal}}) \in C_{\text{free}} \times C_{\text{free}}$  where  $q_{\text{init}}$  is the starting configuration and  $q_{\text{goal}}$  is the goal configuration. This type of graph is known as a roadmap. The procedure for querying it is to add  $q_{\text{init}}$  and  $q_{\text{goal}}$  to the roadmap in the same way sampled configurations are added during the offline phase. Then, a discrete graph search is performed to find a path on the roadmap between the two configurations.

### A. $k$ -PRM\*

Asymptotic optimality is the property that, given enough time, solutions to the path planning problem will almost surely converge to the optimum as defined by some cost function.

*Definition 2 (Asymptotic Optimality in Path Planning):*

An algorithm is *asymptotically optimal* if, for any path planning problem  $(C_{\text{free}}, q_{\text{init}}, q_{\text{goal}})$  and cost function  $c : \Sigma \rightarrow \mathbb{R}_{\geq 0}$  with a robust optimal solution of finite cost  $c^*$ , the probability of finding a solution of cost  $c^*$  converges to 1 as the number of iterations approaches infinity.

Asymptotic optimality is defined only for *robustly feasible* problems as defined in the presentation of  $k$ -PRM\* [12]. Such problems have a minimum clearance around the optimal solution and cost functions with a continuity property.

The method with which neighbors are selected is an important variable in the PRM. In *fully-connected* PRM, all existing samples in the roadmap are considered neighbors, regardless of distance. This results in a highly dense roadmap. The number of edges can be reduced by restricting the maximum distance that two samples are considered neighbors, as in *simple* PRM. The number of edges still grows quadratically with the number of samples in this case. The density of the roadmap can be further reduced by considering only a fixed number of the nearest samples as neighbors, as in *k-nearest* PRM. Although *k-nearest* PRM produces the sparsest roadmaps of the three, it does not provide asymptotic optimality [12]. The properties of these and the following variations are described in Table I.

algorithm	edges	optimal?
complete PRM	$O(n^2)$	asymptotically
simple PRM	$O(n^2)$	asymptotically
$k$ -nearest PRM	$O(kn)$	no
PRM*	$O(n \log n)$	asymptotically
$k$ -PRM*	$O(n \log n)$	asymptotically
SRS	$O(an^{1+\frac{1}{a}})$	asymptotically near
IRS	$O(n \log n)$	asymptotically near

TABLE I  
VARIATIONS OF PRM AND THEIR PROPERTIES.

The PRM\* and k-PRM\* algorithms rectify this by selecting a number of neighbors that is a logarithmic function of the current size of the roadmap. Specifically, for k-PRM\*, the number of nearest neighbors is  $k(n) = k_{\text{PRM}} \log n$ , where  $k_{\text{PRM}} > e(1 + 1/d)$ ,  $n$  is the number of roadmap nodes and  $d$  the dimensionality of the space. Roadmaps constructed by these variations are asymptotically optimal [12]. Nevertheless, the number of neighbors selected for connection attempts still grows with each iteration.

### B. Graph Spanners

A graph spanner, as formalized in the related literature [30], is a sparse subgraph. Given a weighted graph  $G = (V, E)$ , a subgraph  $G_S = (V, E_S \subset E)$  is a  $t$ -spanner of  $G$  if for all pairs of vertices  $(v_1, v_2) \in V$ , the shortest path between  $v_1$  and  $v_2$  in  $G_S$  is no longer than  $t$  times the shortest path between  $v_1$  and  $v_2$  in  $G$ . Because  $t$  specifies the amount of additional length allowed in the solution returned by the spanner, it is known as the *stretch factor* of the spanner.

A simple method for spanner construction, which is a generalization of Kruskal's algorithm for the minimum spanning tree, is given in Algorithm 1 [31]. Instead of accepting only edges that connect disconnected components, this algorithm also accepts edges that add useful cycles. Kruskal's algorithm is recovered by setting  $t$  to a large value, such that no cycle is useful enough to be added.

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#### Algorithm 1 GRAPHSPANNER( $V, E, t$ )

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1: sort  $E$  by non-decreasing weight
2:  $E_S \leftarrow \emptyset, G_S \leftarrow (V, E_S)$ 
3: for all  $(v, u) \in E$  do
4:   if SHORTESTPATH( $V, E_S, v, u$ )  $> t \cdot$ WEIGHT( $v, u$ )
     then
5:      $E_S \leftarrow E_S \cup \{(v, u)\}$ 
6:   end if
7: end for
8: return  $G_S$ 

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The inclusion criteria on line 4 ensures that no edges required for maintaining the spanner property are left out. From the global ordering of the edges of line 1, it has been shown that the number of edges retained is reduced from a potential  $O(n^2)$  to  $O(n)$  [31].

The quadratic number of shortest path queries puts the time complexity into  $O(n^3 \log n)$ ; however, much work has been done to find algorithms that reduce this. Most perform clustering in a preprocessing step and one method uses this idea to reduce the time complexity to  $O(m)$ , where  $m$  is the number of edges [32]. It is this algorithm that was chosen for the sequential approach presented in this work.

A number of alternatives take advantage of the implicit weights of a Euclidean metric space to improve performance. Many of these operate on *complete* Euclidean graphs [33], [34]. They cannot be easily used in  $C$ -spaces with obstacles because obstacles remove edges of the complete graph. Table II lists a range of spanner algorithms that can be applied to weighted graphs along with their properties.

TABLE II  
SPANNER ALGORITHMS AND THEIR PROPERTIES

algorithm	stretch	edges	time
Althöfer et al. [31]	$2a - 1$	$O(n^{1+\frac{1}{a}})$	$O(an^{1+\frac{1}{a}})$
Cohen [35]	$2a + \epsilon$	$O(an^{1+\frac{1}{a}})$	$O(an^{\frac{1}{a}})$
Awerbuch et al. [36]	$64a$	$O(an^{1+\frac{1}{a}})$	$O(mn^{\frac{1}{a}})$
Roditty et al. [37]	$2a - 1$	$O(n^{1+\frac{1}{a}})$	$O(an^{2+\frac{1}{a}})$
Thorup et al. [38]	$2a - 1$	$O(an^{1+\frac{1}{a}})$	$O(amn^{\frac{1}{a}})$
Baswana et al. [32]	$2a - 1$	$O(an^{1+\frac{1}{a}})$	$O(am)$

### III. APPROACH

The central concept proposed in this work is that of asymptotic *near-optimality*. This new property is a relaxation of asymptotic optimality that permits an algorithm to converge to a solution that is within  $t$  times the cost of the optimum.

*Definition 3 (Asymptotically Near-Optimal Path Planning):* An algorithm is *asymptotically near-optimal* if, for any path planning problem  $(C_{\text{free}}, q_{\text{init}}, q_{\text{goal}})$  and cost function  $c : \Sigma \rightarrow \mathbb{R}_{\geq 0}$  that admit a robust optimal solution with finite cost  $c^*$ , the probability that it will find a solution of cost  $c \leq tc^*$  for some stretch factor  $t \geq 1$  converges to 1 as the number of iterations approaches infinity.

The high-level approach is to combine the construction of an asymptotically optimal roadmap with the execution of a spanner algorithm. These two tasks can be performed either sequentially (first find the roadmap, then compute the spanner), or incrementally by interleaving the steps of each task.

#### A. Sequential Roadmap Spanner

An asymptotically optimal roadmap  $G$  generated by k-PRM\* can be used to construct an asymptotically *near-optimal* roadmap  $G_S$  by selectively removing existing edges. Applying an appropriate spanner algorithm to the roadmap accomplishes this and is referred to in this work as the Sequential Roadmap Spanner (SRS) algorithm. Algorithm 2 outlines the operation of the SRS method at a high-level. It first makes a call to the k-PRM\* algorithm, which populates the graph  $(V, E)$ , and then SRS computes the spanner of  $(V, E)$ . A stopping criterion is assumed to be used by k-PRM\*, so that it returns in finite time.

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#### Algorithm 2 SEQUENTIALROADMAPSPANNER( $a$ )

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```

1:  $V \leftarrow \emptyset; E \leftarrow \emptyset$ 
2: k-PRM*( $V, E$ )
3: RANDOMIZEDSPANNER( $V, E, a$ )
4: return  $(V, E)$ 

```

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While any number of spanner algorithms could be used for this purpose, here, a method with linear time complexity with respect to the number of edges is adopted. The graph spanner method employed by SRS is outlined below.

1) *Randomized  $(2a-1)$  Spanner Algorithm [32]:* The input to this approach (Algorithm 3) is the sets of vertices and weighted edges of the roadmap,  $V$  and  $E$ ; and the spanner parameter,  $a$ . The stretch factor  $t$  is a function of the input parameter  $a$ :  $t = (2a - 1)$ . The output is  $E_S$ , the edges of the

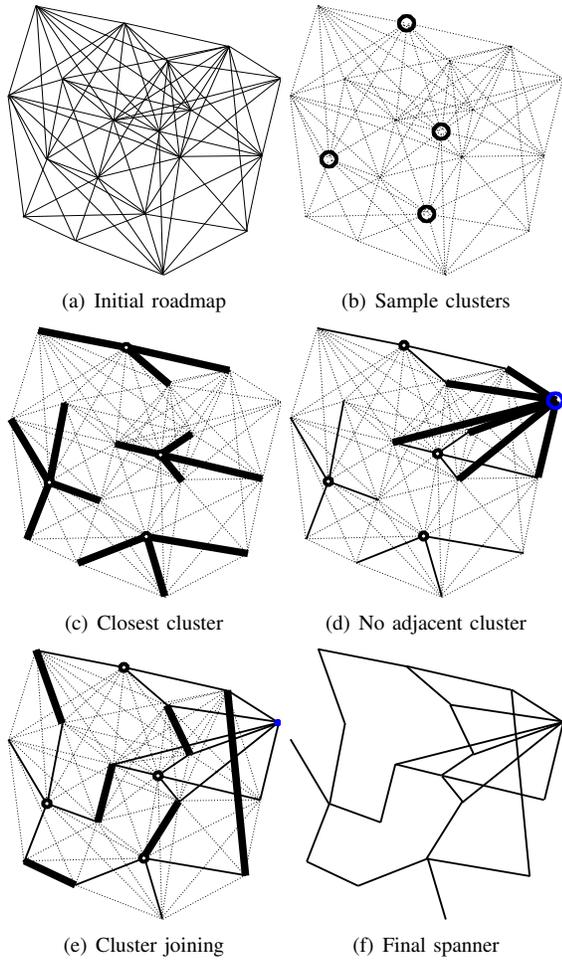


Fig. 2. Illustration of the Randomized  $(2a - 1)$ -Spanner Algorithm for  $a = 2$ .

spanner. There are two parts to the algorithm. First, clusters are formed. Second, the clusters are joined to each other.

A cluster  $c$  is a set of vertices. A clustering  $K_i$  is the set of clusters at iteration  $i$ . Vertices only belong to one cluster in any clustering. Before the first iteration, each vertex is made into its own, singleton cluster to form the initial  $K_0$  clustering. With each successive iteration, the following steps are performed:

- 1) A subset of the previous iteration's clusters are sampled randomly (Figure 2(b)).
- 2) The vertices in the remaining, unsampled clusters are then split up and added to their neighboring clusters (Figure 2(c)).
- 3) Vertices with no adjacent clusters in the *current* clustering are connected to all adjacent clusters from the *previous* clustering (Figure 2(d)).

This is repeated until  $\lfloor \frac{a}{2} \rfloor$  iterations have been performed. In the second part of the algorithm (line 22), the clusters are joined by the shortest edge that connects them to their neighbors (Figure 2(e)).

Two auxiliary functions will be useful in the definition of the  $(2a - 1)$ -spanner algorithm. The function  $E(v, c)$  returns all edges in the set  $E$  from the vertex  $v$  to vertices in cluster  $c$ . Similarly, the function  $E(c, c')$  returns all edges in  $E$

connecting any vertex in cluster  $c$  to any in cluster  $c'$ .

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**Algorithm 3** RANDOMIZEDSPANNER( $V, E, a$ )
 

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1:  $E_S \leftarrow \emptyset$ 
2:  $K_0 \leftarrow \{\{v\} | v \in V\}$ 
3: for  $i \in \{1, \dots, \lfloor \frac{a}{2} \rfloor\}$  do
4:    $K_i \leftarrow \emptyset$ 
5:   for  $c \in K_{i-1}$  do
6:     if UNIFORMRANDOM(0, 1) <  $\|V\|^{-\frac{1}{a}}$  then
7:        $K_i \leftarrow K_i \cup \{c\}$ 
8:        $V_i \leftarrow V_i \cup c$ 
9:     end if
10:  end for
11:  for  $v \in V - V_i$  do
12:     $n_v \leftarrow$  nearest cluster in  $K_i$  to  $v$ 
13:  end for
14:  ADDSPANNEREDGES( $V, V_i, E, E_S, K, n$ )
15:  for  $e \in E$  do
16:     $(v, u) \leftarrow e$ 
17:    if  $v$  and  $u$  are in the same cluster of  $K_i$  then
18:       $E \leftarrow E - \{e\}$ 
19:    end if
20:  end for
21: end for
22: CLUSTERJOINING( $K, E, a, E_S$ )
23: return  $E_S$ 

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The clustering  $K_0$  is initialized (line 2) so that each vertex becomes its own cluster. Lines 5–7 randomly sample the last iteration's clusters with probability  $n^{-\frac{1}{a}}$  (Figure 2(b)). A probability smaller than 1 has the important effect of reducing the number of clusters at each iteration. As a notational convenience,  $V_i$  represents the vertices that belong to sampled clusters at iteration  $i$  (line 8).

Sampled clusters are expanded in lines 11–14. Here, the closest, neighboring, sampled cluster,  $n_v$ , is found for each vertex that is not a member of a sampled cluster. The algorithm shown in Algorithm 4 adds edges to the spanner that connect the new vertices to their clusters.

For those vertices that are adjacent to sampled clusters, lines 3–5 add the shortest edge to the nearest sampled cluster and discards the others. To maintain the spanner property, edges to other, unsampled clusters, that are shorter than this must also be retained (lines 6–12). At the conclusion of each iteration (lines 15–20, in Algorithm 3), intra-cluster edges are removed from  $E$ .

After all  $\lfloor \frac{a}{2} \rfloor$  iterations have completed,  $K_{\lfloor \frac{a}{2} \rfloor}$  contains the final clusters, each of which is a tree rooted at the cluster center.  $E$  now contains only inter-cluster edges, and the task is to decide which of these must be added to the spanner. In Algorithm 5, only the shortest edge to each neighboring cluster is added. To maintain the spanner property, clusters from the previous iteration must also be considered when  $a$  is even (line 4).

In this manner, a sparse roadmap spanner can be constructed from a dense roadmap. Nevertheless, constructing an entire dense roadmap before removing edges can be wasteful. Every

**Algorithm 4** ADDSPANNEREDGES( $V, V_i, E, E_S, K, n$ )

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```

1: for  $v \in V - V_i$  do
2:   if  $v$  has edges incident on  $V_i$  then
3:      $e_v \leftarrow \text{SHORTEST}(E(v, n_v))$ 
4:      $E_S \leftarrow E_S \cup \{e_v\}$ 
5:      $E \leftarrow E - E(v, n_v)$ 
6:     for  $c' \in K_{i-1}$  do
7:        $e_c \leftarrow \text{SHORTEST}E(v, c')$ 
8:       if  $\text{WEIGHT}(e_c) < \text{WEIGHT}(e_v)$  then
9:          $E_S \leftarrow E_S \cup \{e_c\}$ 
10:         $E \leftarrow E - E(v, c')$ 
11:       end if
12:     end for
13:      $n_v \leftarrow n_v \cup \{v\}$ 
14:   else
15:     for  $c \in K_{i-1}$  do
16:        $E_S \leftarrow E_S \cup \text{SHORTEST}(E(v, c))$ 
17:        $E \leftarrow E - E(v, c)$ 
18:     end for
19:   end if
20: end for

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**Algorithm 5** CLUSTERJOINING ( $K, E, a, E_S$ )

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1: if  $a$  is odd then
2:    $i \leftarrow \lfloor \frac{a}{2} \rfloor$ 
3: else
4:    $i \leftarrow \lfloor \frac{a}{2} \rfloor - 1$ 
5: end if
6: for  $c \in K_{\lfloor \frac{a}{2} \rfloor}$  do
7:   for  $c' \in K_i$  do
8:      $E_S \leftarrow E_S \cup \text{SHORTEST}(E(c, c'))$ 
9:   end for
10: end for

```

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edge in the dense roadmap has been checked for collisions, but many of these will be discarded by the spanner algorithm. In the next section, an incremental spanner algorithm is described, which accepts or rejects edges as they are considered for addition to the roadmap and before collision checking is performed. It sacrifices asymptotic time complexity for a reduction in the number of collision checks, which is frequently the most expensive operation in motion planning.

*B. Incremental Roadmap Spanner*

The Incremental Roadmap Spanner (IRS, Algorithm 6) takes the idea of the SRS one step further. Here, roadmap and spanner construction are interleaved. When the roadmap algorithm adds an edge, the spanner algorithm can reject it before collision detection is performed. Before discussing the implementation of the algorithm, some subroutines must be defined for the particular  $C$ -space being worked on:

SAMPLEFREE uniformly samples a random configuration in  $C_{\text{free}}$ .

NEAR( $V, v, k$ ) returns the  $k$  configurations in set  $V$  that are nearest to configuration  $v$  with respect to a metric function. This can be implemented as a linear search through the

members of  $V$  or as something more involved, such as a nearest neighbors search in a  $kd$ -tree [39].

COLLISIONFREE( $v, u$ ) detects if there is a path between configurations  $v$  and  $u$  in  $C_{\text{free}}$ . Generally, a local planner plots a curve in  $C$  from  $v$  to  $u$ . Points along this curve are tested for membership in  $C_{\text{free}}$  and if any fail, the procedure returns false.

STOPPINGCRITERION determines when to stop looking for a solution. Some potential stopping criteria are:

- a solution of sufficient quality has been found
- allotted time or memory have been exhausted
- enough of the  $C$ -space has been covered

or some combination of these or other criteria.

WEIGHT( $v, u$ ) returns the positive edge weight of  $(v, u)$ . In the context of motion planning, the weight of an edge is frequently the cost of moving the robot from configuration  $v$  to configuration  $u$  along the curve provided by a local planner.

SHORTESTPATH( $V, E, v, u$ ) returns the cost of the shortest path between  $v$  and  $u$ . Note that the actual shortest path cost is not required, just whether it is larger than  $t$  times the weight of the edge  $(v, u)$ . Instead of naively applying a full graph search, a length limited variation of A\* search can be employed. In this variation, the search is aborted if a node with a cost value  $f > t \times w(v, u)$  is expanded. When this happens, it is known that there is no path from  $v$  to  $u$  with an acceptable cost. Additionally, edges that connect two disconnected components can be added without doing any kind of search because the shortest path cost in this case is infinite.

**Algorithm 6** INCREMENTALROADMAPSPANNER( $t$ )

---

```

1:  $V \leftarrow \emptyset, E \leftarrow \emptyset$ 
2: while !STOPPINGCRITERION() do
3:    $v \leftarrow \text{SAMPLEFREE}()$ 
4:    $V \leftarrow V \cup \{v\}$ 
5:    $k \leftarrow \lceil e \cdot (1 + \frac{1}{d}) \log(\|V\|) \rceil$ 
6:    $U \leftarrow \text{NEAR}(V, v, k)$ 
7:   sort  $U$  by non-decreasing distance from  $v$ 
8:   for all  $u \in U$  do
9:     if  $\text{SHORTESTPATH}(V, E, v, u) > t \cdot \text{WEIGHT}(v, u)$  then
10:      if  $\text{COLLISIONFREE}(v, u)$  then
11:         $E \leftarrow E \cup \{(v, u)\}$ 
12:      end if
13:    end if
14:  end for
15: end while
16: return  $(V, E)$ 

```

---

First, the roadmap  $G = (V, E)$  is initialized to empty (line 1). Then, it iterates until STOPPINGCRITERION returns true (line 2). For each iteration, SAMPLEFREE is called and returns  $v \in C_{\text{free}}$  (line 3). The sampled vertex is added to the roadmap (line 4). The number of nearest neighbors is calculated (line 5). The set  $U$  of the  $k$ -nearest neighbors of  $v$  are found by calling NEAR( $V, v, k$ ) (line 6). If NEAR does not already return  $U$  ordered by distance from  $v$ , then it must be sorted on line 7. For each potential edge connecting  $v$  to

a neighbor in  $U$ , the inclusion criteria must be met before it is added to the roadmap. First, if a path exists between  $v$  and  $u$  with cost less than  $t$  times the weight of  $(v, u)$  then that edge can be rejected because it contributed little to path quality (line 9). Second, the edge must be checked for collision (line 10), as it is typically the case in the PRM framework. If the local planner does not succeed in finding a curve in  $C_{\text{free}}$  the edge is rejected. If the edge passes both inclusion tests, it is added to the roadmap (line 11). Then, the next iteration is started.

A notable departure that IRS makes from Algorithm 1 is that the edges are not ordered globally. This ordering is not required to preserve solution quality, however, as will be shown later. A *local* ordering of potential edges is performed on line 7. This does not affect the theoretical bounds on solution quality, but can be seen as a heuristic that improves the sparsity of the final roadmap.

Since the spanner property is tested before the collision check, many expensive collision checks can be avoided. This property greatly improves running time for practically sized roadmaps.

#### IV. ANALYSIS

The properties of SRS and IRS are formally described in this section.

##### A. Asymptotic Near-Optimality

*Theorem 1:* SRS is asymptotically near-optimal.

*Proof:* Asymptotic Near-Optimality (Definition 3) differs from Asymptotic Optimality (Definition 2) only in that the solution costs must converge to within  $t$  times the optimal cost instead of the actual optimal cost.

It is known that solutions on roadmaps produced by  $k$ -PRM\* converge to the optimal costs [12]. A  $t$ -spanner of a such a roadmap is guaranteed to have solutions within  $t$  times the cost of those in the original roadmap. SRS produces  $t$ -spanners of roadmaps constructed by  $k$ -PRM\*. Therefore, solutions on roadmaps produced by SRS converge to within  $t$  times the cost of optimal, as the construction of the original roadmap by  $k$ -PRM\* goes to infinity. ■

*Theorem 2:* IRS is asymptotically near-optimal.

*Proof:* This proof also relies on the asymptotic optimality of  $k$ -PRM\* on which IRS is based. The proposed technique has two differences from  $k$ -PRM\*.

First, on line 7, the potential neighbors of a newly added vertex are ordered by non-decreasing distance from the new vertex. This would have no effect on the asymptotic optimality of  $k$ -PRM\* because edges are rejected based solely on collisions with obstacles. The order that edges are tested for collisions has no effect on those tests. The second difference, on line 9, adds an additional acceptance criterion. This has the effect of making the edges in a roadmap output by IRS a subset of those output by  $k$ -PRM\*.

Consider a pair of roadmaps,  $G = (V, E)$  returned by  $k$ -PRM\* and  $G_S = (V, E_S \subset E)$  returned by IRS. For each edge  $(v, u)$  in  $E/E_S$ , there was a path from  $v$  to  $u$  with a cost less than  $t$  times the weight of  $(v, u)$ . This invariant is

enforced by line 9. The shortest path  $\sigma$  in  $G$  between any two points  $a, b \in V$  has cost  $c^*$ . This path may contain edges that are in  $E$  but not in  $E_S$ . For each of these edges  $(v, u)$ , there exists an alternate path in  $G_S$ , with cost  $c_{(v,u)} \leq t \cdot w(v, u)$ . Therefore, there is a path  $\sigma_S$  between  $a$  and  $b$  in  $G_S$  with cost  $\sum_{(v,u) \in \sigma_S} c_{(v,u)} \leq t \cdot c^*$ . In other words, since each detour is no longer than  $t$  times the cost of the portion of the optimal path it replaces, the sum cost of all of the detours will not exceed  $t$  times the total cost of the optimal path. In this way, as construction time goes to infinity, the roadmap returned by IRS becomes near-optimal with probability 1. ■

##### B. Time Complexity

The time complexity analysis of SRS and IRS can be understood in relation to the complexity of  $k$ -PRM\*. For each of the  $n$  iterations of  $k$ -PRM\*, a nearest neighbors search and collision checking must be performed. An  $\epsilon$ -approximate nearest neighbor search can be done in  $\log n$  time producing  $k(n) = \log n$  neighbors. The edge connecting the current iteration's sample to each of these neighbors must be checked for collision at a cost of  $\log^d p$  time, where  $p$  is the number of obstacles. So, the total running time of  $k$ -PRM\* is  $O(n \cdot (\log n + \log n \cdot \log^d p))$ , which simplifies to  $O(n \cdot \log n \cdot \log^d p)$ .

- Time complexity breakdown for each of  $n$  iterations of  $k$ -PRM\* (ignoring constant time operations):
  - NEAR ( $\epsilon$ -approximate):  $\log n$
  - For each  $\log n$  neighbors:
    - \* COLLISIONFREE:  $\log^d p$

The time complexity of SRS is adding to the complexity of  $k$ -PRM\* the complexity of the graph-spanner algorithm ( $O(kn)$ ). The linear complexity of the graph-spanner algorithm is dominated by that of  $k$ -PRM\*. Thus, overall the complexity of SRS is  $O(n \cdot \log n \cdot \log^d p)$ .

- Time complexity breakdown for each of  $n$  iterations of IRS (ignoring constant time operations):
  - NEAR ( $\epsilon$ -approximate):  $\log n$
  - Neighbor ordering:  $\log n \cdot \log \log n$
  - For each  $\log n$  neighbors:
    - \* COLLISIONFREE:  $\log^d p$
    - \* SHORTESTPATH  $> t \cdot \text{WEIGHT}$ :  $t^d \log n \cdot \log(t^d \log n)$  (average case)

For IRS, two additional steps are performed. At every iteration,  $k = \log n$  neighbors must be sorted by their distance to the sampled vertex. Many implementations of NEAR return the list of neighbors in order of distance, in which case the cost of this operation can be zero. If this is not the case, however, the cost of sorting these  $k$  neighbors is  $O(k \log k)$ , and since  $k = \log n$ , the final result is:  $O(\log n \cdot \log(\log n))$ .

A shortest path search must also be performed for each potential edge. If done across the entire roadmap, this has a cost of  $O(m \log n) = O(n \log^2 n)$ , where  $m$  is the number of edges and  $m = n \log n$ , since each node is connected to at most  $\log n$  neighbors. The shortest path algorithm will expand only the vertices with path cost from  $v$  that is lower than  $t \cdot w(v, u)$ . This is due to the fact that the algorithm requires only knowledge

about the existence of a path between  $v$  and  $u$  shorter than this bound. When  $t = 1$ , the number of nodes expanded by such a search is, at most,  $k(n) \in O(\log n)$ . Assuming a uniform sampling distribution, the expected number of nodes that may be expanded when  $t > 1$  is proportional to  $t^d \log n$  (based on the volume of a  $d$ -dimensional hyper-sphere). This brings the *expected* time complexity of IRS to

$$O\left(n \cdot \left(\log n + \log n \cdot \log \log n + \log n \cdot (\log^d p + t^d \log n \cdot \log(t^d \log n))\right)\right)$$

which simplifies to:  $O(n \cdot \log n \cdot t^d \log n \cdot \log(t^d \log n))$ , or, for fixed  $t$  and  $d$ :

$$O(n \cdot \log^2 n \cdot \log \log n)$$

which is asymptotically slower than  $k$ -PRM\*. However, as will be shown experimentally, the very large constants involved in collision checking, which is avoided by IRS, makes IRS a computationally efficient alternative, at least for roadmaps of practical size.

### C. Size Complexity

The number of edges in a spanner produced by Algorithm 3 is  $O(an^{1+\frac{1}{a}})$  which also bounds the number of edges produced by SRS.

IRS is inspired by Algorithm 1, which produces  $O(n)$  edges (with constants related to  $t$  and  $d$ ). This bound does not hold for IRS itself. Without a global ordering of edges, there is no guarantee that the minimum spanning tree is contained within the spanner. Because of this, the space complexity cannot be bounded lower than that provided by  $k$ -PRM\*, which is  $O(n \log n)$ . The experimental results, however, suggest that the dominant term is linear as the number of edges added at each iteration appear to converge to a constant.

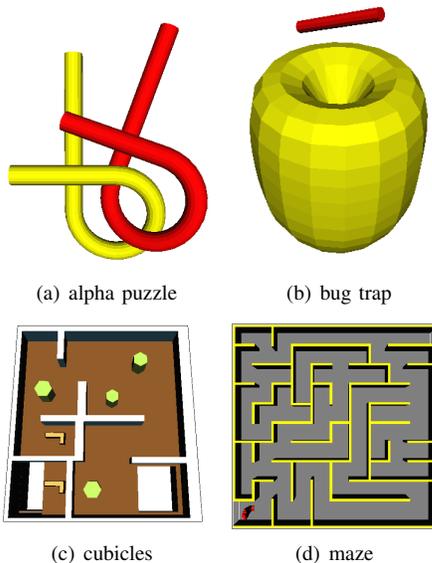


Fig. 3. Environments used in the experiments with example configurations for the robot.

## V. EVALUATION

All experiments were executed using the Open Motion Planning Library (OMPL) [40] on 2GHz processors with 4GB of memory. Four representative environments were chosen from those distributed with OMPL.

*alpha puzzle*: The  $C_{\text{free}}$  is a subset of  $SE(3)$  and is highly constrained in this classical motion planning benchmark (Figure 3(a)).

*bug trap*: A rod shaped robot must orient itself to fit through the narrow passage and escape a three-dimensional version of the classical bug trap ( $SE(3)$  - Figure 3(b)).

*cubicles*: Two floors of loosely constraining walls and obstacles must be navigated ( $SE(3)$  - Figure 3(c)).

*maze*: A complex environment in a lower dimensional ( $SE(2)$ ) space (Figure 3(d)).

For each of these environments and stretch factor combination, 10 roadmaps with 50,000 vertices each were generated. On each of these roadmaps, 100 random start and goal pairs were queried over the course of construction and various qualities of the resulting solutions were measured. An additional  $SE(3)$  environment (*apartment*) was considered in Fig. 4 as a representative case for high collision detection costs.

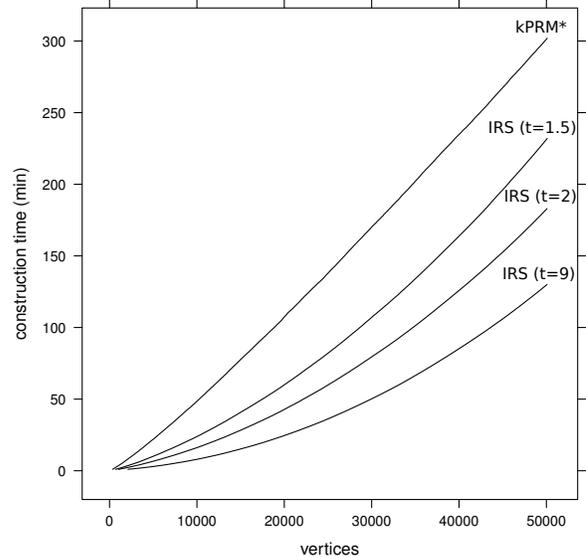


Fig. 4. Construction time and roadmap density for  $k$ -PRM\* and IRS in the apartment environment (shown below). When the cost to perform collision detection is high, as in the complex apartment environment, practically sized roadmap spanners can be constructed faster because fewer edges must be checked for collision.

### A. Construction Time

Although the expected asymptotic time complexity of IRS is worse than that of  $k$ -PRM\*, the large constants involved in collision detection dominate the running time in many cases. Since IRS reduces the number of collision checks required at the cost of including a call to a graph search algorithm (due to the check on line 9), running time can be reduced for higher stretch factors. This is shown in Figure 4, where a stretch factor of  $t = 2$  allows IRS to construct a 50,000 node roadmap in under two-thirds the time of  $k$ -PRM\*. The diminishing returns shown for higher stretch factors reflect the larger area of the graph that must be searched for shortest paths. Construction times for the other experiments are shown in Figure 5. Whether a time savings is achieved is dependent on features of the environment that affect collision checking such as the density of the obstacles and how expensive each collision check is to perform.

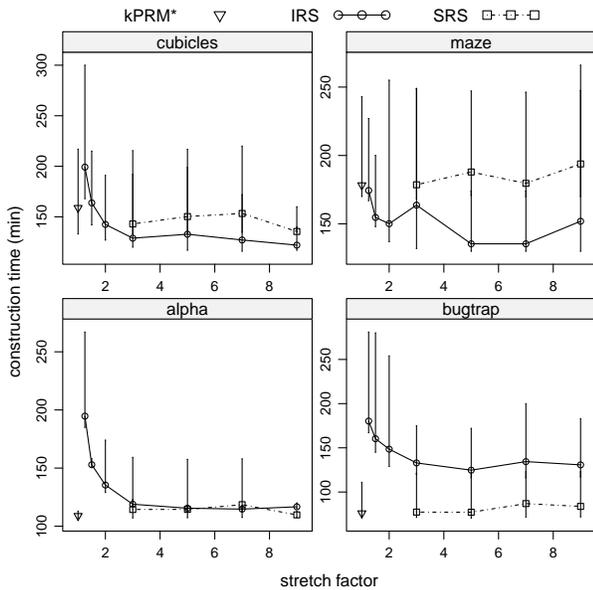


Fig. 5. The mean construction times of 50,000 node roadmaps for each environment. Error bars indicate the minimum and maximum times over 10 runs.

### B. Space Requirements

The reduction in edges by the roadmap spanners is shown in Figure 6. While each roadmap contains the same number of vertices (50,000), the space required for connectivity information is reduced by up to 85%. Although SRS has a lower asymptotic bound for the number of edges produced, IRS shows much better results in practice, for the same stretch factor.

Figure 7 shows the number of edges added to a roadmap as the number of nodes grows for both  $k$ -PRM\* and IRS. In these environments, the linear term appears to dominate even though the growth is bounded by  $O(n \log n)$ .

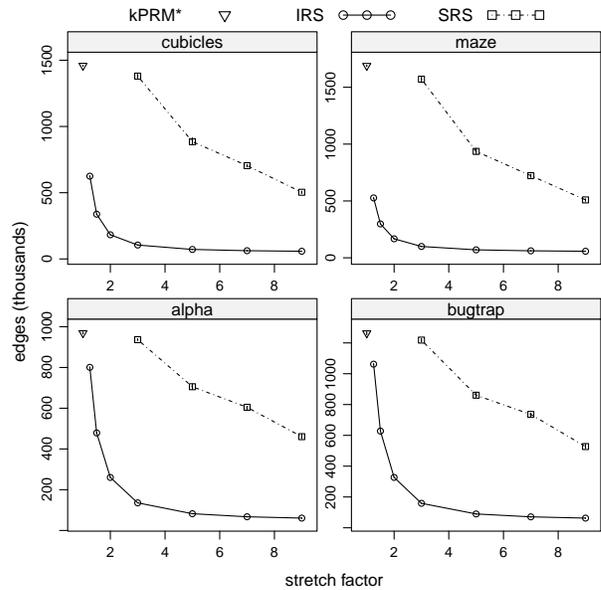


Fig. 6. The mean roadmap density of 10 different 50,000 node roadmaps for each environment. The minimum and maximum values are within the size of the icons.

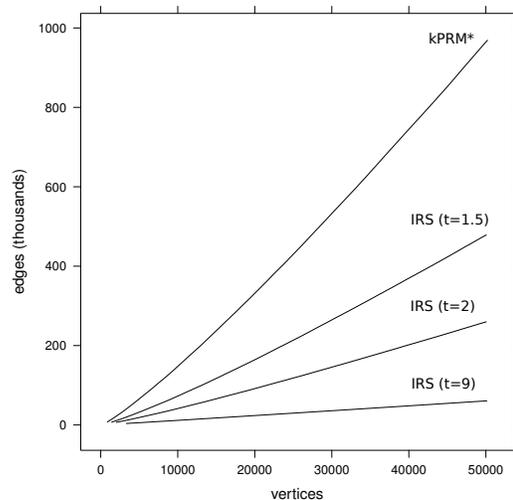


Fig. 7. The density growth for a roadmap constructed in the alpha environment. The reduction in the number of edges increases with stretch factor and grows approximately linearly with the number of vertices.

### C. Solution Quality

In Figure 8, path quality is measured by querying a roadmap with 100 random start and goal configurations. The lengths of the resulting paths increase as the number of edges in the spanner is reduced. It is important, however, to note that for these random starting and goal configurations, the average extra cost is significantly shorter than the worst case guaranteed by the stretch factor.

The worst degradation happens for short paths, where taking a detour of even a single vertex can increase the path length by a large factor. Path quality degradation in IRS is plotted in Figure 9 as a function of its length in a roadmap generated by

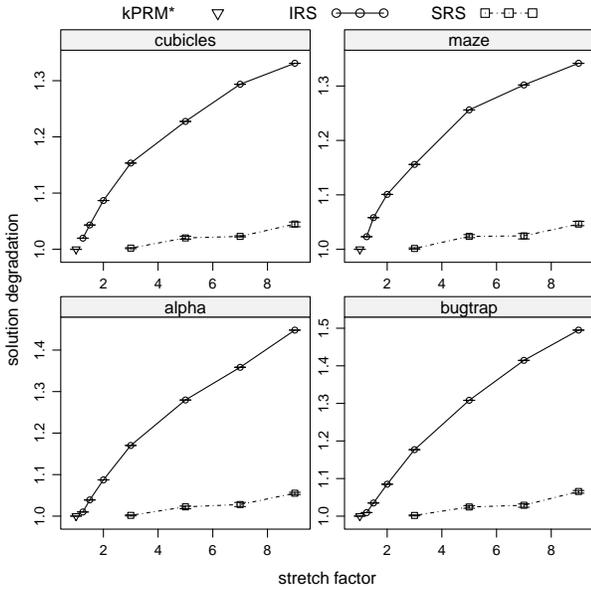


Fig. 8. Mean solution length for 100 random query pairs on 10 different 50,000 node roadmaps compared to the  $k$ -PRM\* outcome. Path length increases when the stretch factor increases. The increase is much less than the theoretical guarantee.

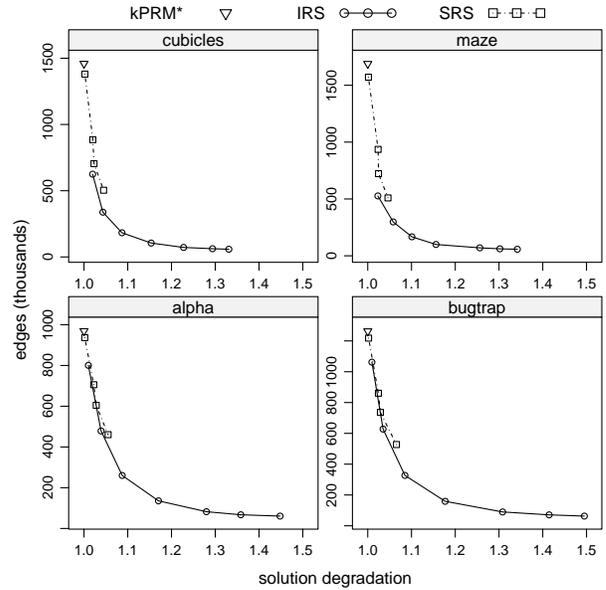


Fig. 10. Trade-off between solution length and roadmap density averaged over 100 random query pairs on 10 different 50,000 node roadmaps for each environment. Each data point for IRS and SRS represents a different stretch factor from 1.25 to 9. Except for the alpha environment, IRS dominates SRS in these two performance measures.

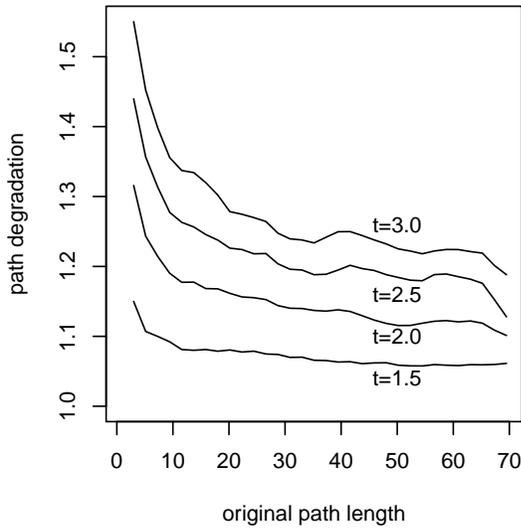


Fig. 9. Mean path length degradation for different stretch factors as a function of the path length in the original roadmap in the bug-trap environment with 5,000 vertices. All shortest paths starting from 10 random vertices in 5 different roadmaps were measured.

$k$ -PRM\*. All shortest paths to 10 random vertices are averaged over 5 different roadmaps. In Figures 10 and 11, the trade-off between roadmap density and path quality is compared. It can be observed from these figures that IRS dominates both SRS and a naïve approach where random edges are removed. Table III provides the mean roadmap construction time in minutes for the algorithms in the different environments.

Overall, the results show that the SRS is not able to provide a large reduction in edges for low stretch values. For example, in one environment it was only able to reduce the edge count

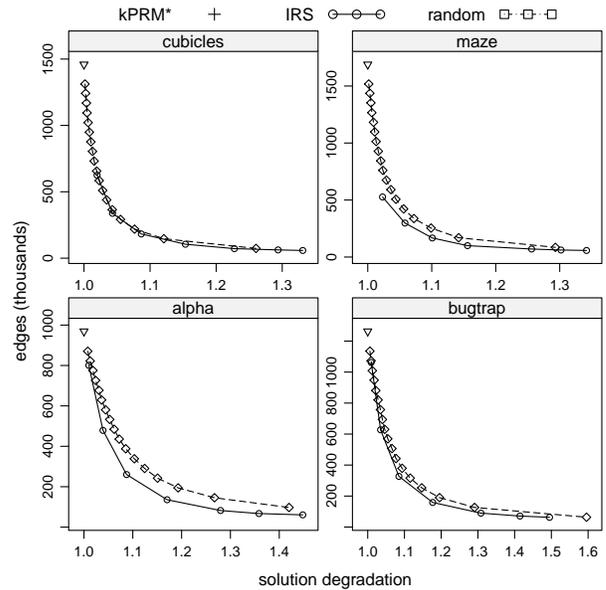


Fig. 11. Trade-off between solution length and roadmap density averaged over 100 random query pairs on 10 different 50,000 node roadmaps for each environment. Each data point for IRS represents a different stretch factor from 1.25 to 9. Each data point for the “random” method represents the probability an edge being removed from 5% to 90%. IRS dominates this random method in these two performance measures.

by 15.3% for a stretch factor of 3. For higher stretch factors, e.g., up to 17, most of the edges (77% to 85%) can be removed by SRS, while increasing mean path length by a small amount (10% to 25%). The Incremental Roadmap Spanner (IRS) described in Algorithm 1 was able to achieve a 70.5% edge reduction for relatively small stretch factors.

algorithm	alpha	bugtrap	cubicles	maze
k-PRM*	109	76	159	178
SRS	114	81	146	185
IRS	136	145	145	152
random	113	80	142	171

TABLE III

MEAN CONSTRUCTION TIME (MINUTES) FOR ALGORITHMS IN DIFFERENT ENVIRONMENTS.

#### D. Roadmap Search Time

The time it took to perform an A\* search on the resulting roadmap was also measured. The reported value does not include the time it takes to connect the start and goal configurations. As shown in Fig. 12, removing edges from the roadmaps reduces the query resolution time by up to 70%.

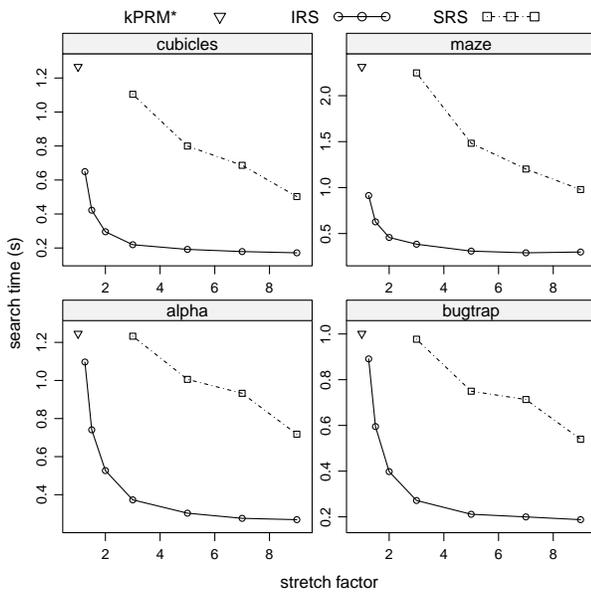


Fig. 12. Mean search time for 100 random query pairs on 10 different 50,000 node roadmaps for each environment. Higher stretch factors produce roadmaps with fewer edges that can be searched more quickly.

While reduced online query resolution time is presented here as a separate benefit, it is dependent on the size of the generated roadmap. Larger roadmaps generally result in longer search times. Experiments suggest that search time has an approximately linear relationship to the number of edges in roadmaps of these environments. This is illustrated by Fig. 13 and explains the similarity between Figs. 6 and 12.

#### E. Effects of Smoothing

For each query, a simple approach for path smoothing was tested. Non-consecutive vertices on the path that were near each other were tested for potential collision-free connectivity. If they could be connected, then the path is shortened by removing the intervening vertices. This is a greedy and local method for smoothing, but it can be executed very quickly and produces impressive results. In Figure 14, the smoothing factor is reported as the ratio of the smoothed cost to the unsmoothed cost. The time taken to smooth the solutions was between

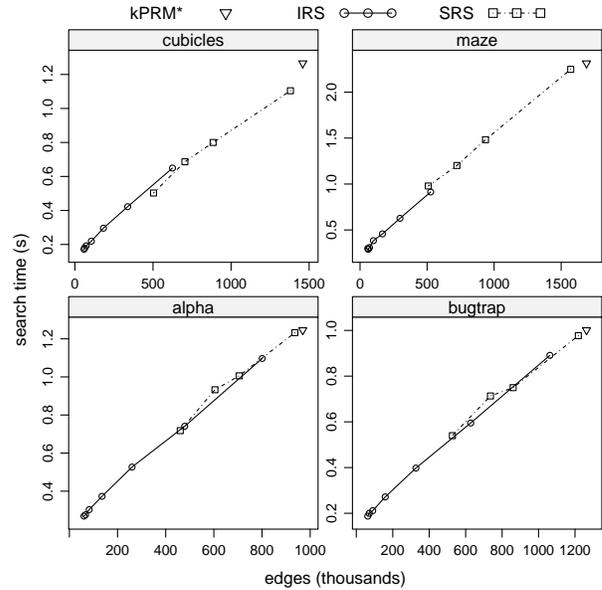


Fig. 13. In these environments, (and in the environment shown in Fig. 4) roadmap search time has an approximately linear relationship to the density of the roadmap.

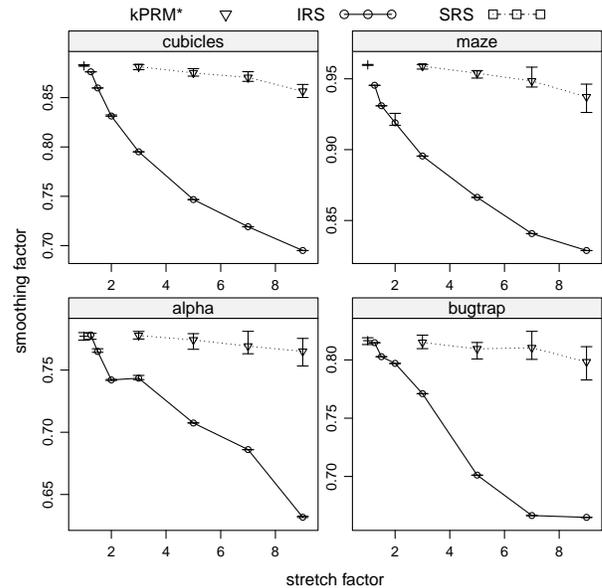


Fig. 14. The ratio of the cost of the smoothed solution to that of the unsmoothed one for 100 random query pairs on 10 different 50,000 node roadmaps for each environment.

0.01 and 0.7 seconds. A smoothing factor of less than one indicates that there was some improvement due to smoothing. Note that k-PRM\* has yet to converge to the optimal solution so applying smoothing still improves the solution path. The solutions from roadmap spanners constructed with higher stretch factors showed more improvement after smoothing, reducing their cost relative to the smoothed k-PRM\* solutions.

## VI. DISCUSSION

### A. Selecting a Stretch Factor

The stretch factor parameter  $t$  should be selected based on the requirements of the system at hand. When path quality is of the utmost importance, a low value close to 1 should be used. For tasks that can afford lower path quality, but need to reduce other metrics such as preprocessing time, roadmap size, or online search time, a higher value should be used.

If preprocessing time is not an issue, then the parameter space can be searched until the highest value for  $t$  is found that produces a roadmap that fits into the space or search time allowed. Some tasks will have a natural method for optimizing the stretch factor. Define the total cost of completing a motion planning query as the time to search the roadmap ( $a$ ) plus the time it takes to travel the path ( $b$ ). Increasing  $t$  will decrease  $a$ , but increase  $b$ . Decreasing  $t$  will have the opposite affect. Then the stretch factor  $t$  can be optimized to minimize the worst case for  $a + b$ .

If limits on detour length relative to a known optimum can be defined, then the stretch factor can be chosen directly. For example, if a path that can be followed by a robot is twice as long as another, the robot can be prevented from taking the undesirable path by setting  $t$  to less than 2.

### B. Path Quality Guarantees vs. Actual Average Path Quality

The path quality guarantees in this work are based on the worst possible situation where every edge along an optimal path has been replaced by a detour that is  $t$ -times longer. While this is possible, it has a very low probability of happening for any query that must traverse a significant portion of the roadmap. For very short paths, that will utilize fewer roadmap edges, it is more likely. Nevertheless, the fact that some edges have, necessarily, not been removed pushes the average path degradation lower than the stretch factor limit, especially for queries with longer optimal paths. A practical measure of roadmap path quality might be the average degradation over all possible paths. In general, this is difficult to predict before actually constructing a roadmap, but may be possible to calculate for environments with simple cost functions and no obstacles.

### C. Applicability to Problems with Arbitrary Cost Functions

In order to guarantee near optimal solutions, the cost function used in the methods presented here must be Lipschitz continuous. Such a function is relatively easy to construct for kinematic systems. If no guarantees are required then an arbitrary cost function may be employed. The practical benefits of faster construction and search time, smaller roadmap than  $k$ -PRM\* and better path quality than  $k$ -PRM should still be present with most cost functions. If the cost function does not create a metric space, then a graph search algorithm other than A\* should be used. Any work on asymptotically optimal planners that operate with arbitrary cost functions move in an orthogonal direction and should be possible to integrate with the current work on roadmap spanners.

## VII. CONCLUSION

This work shows that it is practical to compute sparse roadmaps in  $C$ -spaces that guarantee asymptotic near-optimality. These roadmaps have considerably fewer edges than roadmaps with asymptotically optimal paths, while resulting in small degradation in path quality relative to the asymptotically optimal ones. The stretch factor parameter provides the ability to tune this trade-off. The experiments confirm that roadmaps with low stretch factors have high path quality but are denser.

The current approach removes only roadmap *edges*. It is also the case, however, that nodes of the roadmap are redundant for the computation of near-optimal paths. Consecutive work to this one is investigating how to remove nodes so that the quality of a path answering a query in the continuous space is guaranteed not to get worse than a stretch factor [41]–[43].

It is important to study the type of near-optimality guarantees that can be provided in finite time, as in practice a stopping criterion is employed to stop sampling-based algorithms. Furthermore, identifying the expected path quality degradation of a roadmap spanner will be a better indication of practical performance. Another direction to consider is to evaluate these algorithms in higher-dimensional challenges, such as planning of articulated robots. The current experiments do not trend in a way that would suggest that problems would arise and the theoretical analysis covers these challenges.

Finally, it is important to study the relationship of roadmap spanners with methods that guarantee the preservation of the homotopic path classes in the  $C$ -space [10]. Intuitively, homotopic classes tend to be preserved by the spanner because the removal of an important homotopic class will have significant effects in the path quality.

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