Approximating Nash Equilibrium Via Multilinear Minimax

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• We describe Alternate bounds via scaling.

Two Person Zero-Sum Game - Mixed Strategy

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Let

$$S_k = \{u \in \mathbb{R}^k : \sum_{i=1}^{k} u_i = 1, u_i \geq 0\},$$

unit simplex.

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saddle point property.

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Khachiyan (1979), proved linear programming is in P.

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If player 1 reveals p_*^1 as his choice, player 2's best choice is p_*^2.
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Bimatrix Nash Equilibrium can be computed by Lemke-Howsen algorithm - known to be exponential in worst-case.

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von Neumann Minimax is a special case of Nash's Theorem. Algorithmically, von Neumann Equilibrium is computable polynomially, whereas NE is not known to be in P.

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A generalization of von Neumann minimax - a theorem of independent interest.

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Thus solution of MMR, p^**, is a solution, where at least one player's payoff is better than payoff in any Nash equilibrium.*

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(V): More generally, the results extend to any number of players.

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Given $p = (p^1, \ldots, p^t) \in \Delta_n$, the *value* of A at p is:

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$$\max_{x^i \in S_{n_i}} A_i[p_*^1, \dots, p_*^{i-1}, x^i, p_*^{i+1}, \dots, p_*^t] = A_i[p_*].$$

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Analogous to bilinear case (von Newumann minimax), we represent LHS and RHS of (1) as a pair of primal-dual linear programs.

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$$\sigma_t = \frac{1}{t} \sum_{i=1}^{t} A_i[p_*] \leq \frac{1}{t x^*_{min}} A_0[x^*, p^*].$$

(i.e. we get bound on average Nash payoffs)

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In particular, (x^*, p^*) corresponds to $d = (1/t, \dots, 1/t)^T \in \mathbb{R}^t$.

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Thus if $d' > 0$, d induces new MMR and bound for NE.

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THANK YOU!