

Approximating Nash Equilibrium Via Multilinear Minimax

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- We describe **Alternate bounds via scaling**.

Two Person Zero-Sum Game - Mixed Strategy



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Let

$$S_k = \{u \in \mathbb{R}^k : \sum_{i=1}^k u_i = 1, u_i \geq 0\},$$

unit simplex.

Von Neumann Minimax Theorem

Theorem

Given $m \times n$ real $A = (a_{ij})$, there exists $(x_*, y_*) \in S_m \times S_n$ such that

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saddle point property.

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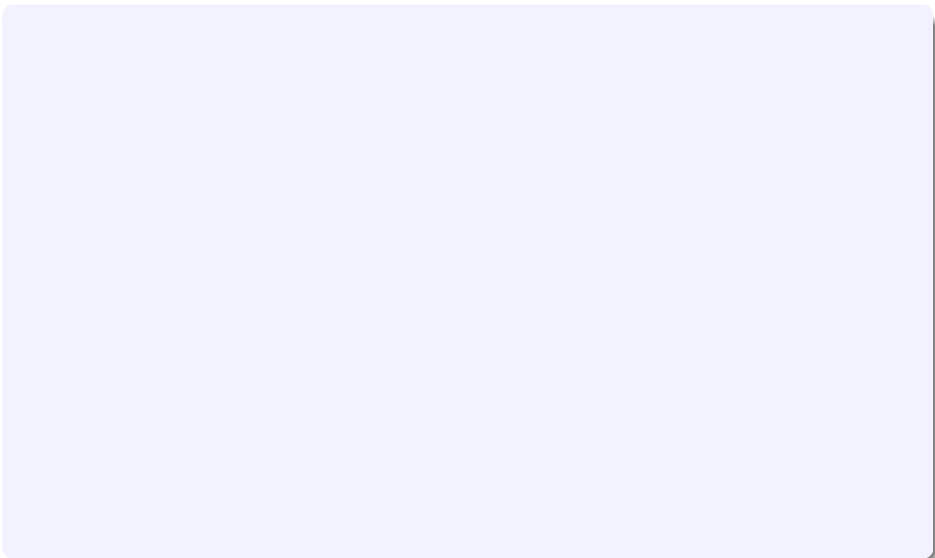
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Khachiyan (1979), proved linear programming is in P.

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$$\max_{p^2 \in S_{n_2}} A_2[p_*^1, p^2] = A_2[p_*^1, p_*^2] = A_2[p_*].$$

If player 1 reveals p_*^1 as his choice, player 2's best choice is p_*^2 .

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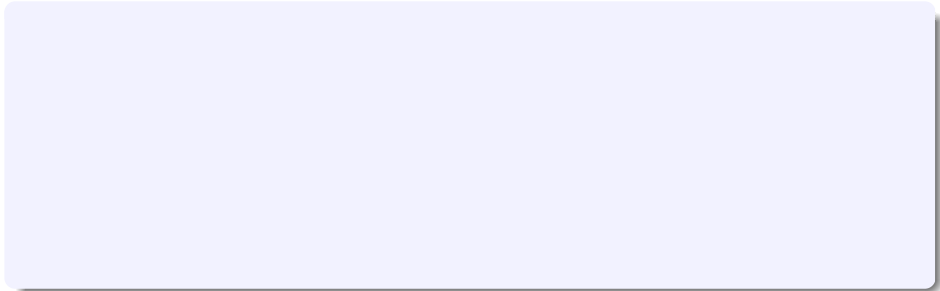
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Bimatrix Nash Equilibrium can be computed by Lemke-Howson algorithm - known to be exponential in worst-case.

Minimax as Special Case of Nash Equilibrium



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von Neumann Minimax is a special case of Nash's Theorem. Algorithmically, von Neumann Equilibrium is computable polynomially, whereas NE is not known to be in P.

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$A_0[x, p]$ “relates” the players’ payoffs.

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A generalization of von Neumann minimax - a theorem of independent interest.

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In particular, from the above it follows at least for some $i = 1, 2$,

$$A_i[p_*] \leq A_i[p^*].$$

Thus solution of MMR, p^* , is a solution, where at least one player's payoff is better than payoff in any Nash equilibrium.

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- (V): More generally, the results extend to any number of players.

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Given $p = (p^1, \dots, p^t) \in \Delta_n$, the *value* of A at p is:

$$A[p] = A[p^1, \dots, p^t] = \sum_{I \in N} a_I p_{i_1}^1 \times \dots \times p_{i_t}^t.$$

Multidimensional Matrix - First Some Notations



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Analogous to bilinear case (von Neumann minimax), we represent LHS and RHS of (1) as a pair of primal-dual linear programs.

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Let $x_{\min}^ = \min\{x_i^* : i = 1, \dots, t\}$. If $x_{\min}^* > 0$, and $A_i[p_*] \geq 0$ for all $i = 1, \dots, n$, then*

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$$\sigma_t = \frac{1}{t} \sum_{i=1}^t A_i[p_*] \leq \frac{1}{tx_{\min}^*} A_0[x^*, p^*].$$

(i.e. we get bound on average Nash payoffs)

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In particular, (x^*, p^*) corresponds to $d = (1/t, \dots, 1/t)^T \in \mathbb{R}^t$.

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Thus if $d' > 0$, d induces new MMR and bound for NE.

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- New applications of MMR in Game Theory, CS, Economics and more.

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THANK YOU!