Optimum
Joint Detection & Estimation
Application to MIMO Radar

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Outline

- Joint Detection and Estimation: problem formulation
- Optimum one- and two-step strategies
- MIMO radar
- Application of the two-step scheme
- Simulations
Joint Detection and Estimation

The problem: For a finite sequence of samples $X = [x_1, \ldots, x_n]$ we assume the following two hypotheses:

$$H_0 : X \sim f_0(X)$$
$$H_1 : X \sim f_1(X|\theta)$$

Detection: Given the data vector $X$, decide between the two hypotheses

Estimation: Every time there is a decision in favor of $H_1$ estimate $\theta$
If a target enters the operational space of our radar, we would like to **detect** it.

Once a target is detected we would also like to **estimate** its position, speed...
Detection problem (Neyman-Pearson)

We are given a data vector $X = [x_1, \ldots, x_n]$ for which we assume the following two hypotheses:

\[ H_0 : \quad X \sim f_0(X) \]
\[ H_1 : \quad X \sim f_1(X) \]

Use $X$ to select $D = H_0$ or $H_1$

False alarm $P_0(D=H_1) \leq \alpha, \quad 0 < \alpha < 1$

Maximize $P_1(D=H_1)$

Minimize $P_1(D=H_0)$
\( H_0 : \ X \sim f_0(X) \)
\( H_1 : \ X \sim f_1(X|\theta), \ \pi(\theta) \)

False alarm \( P_0(D = H_1) \leq \alpha, \ 0 < \alpha < 1 \)

Minimize \( P_1(D = H_0) \)

\[
\int \frac{f_1(X|\theta)\pi(\theta)d\theta}{f_0(X)} \quad \frac{H_1}{H_0} \quad t
\]

\( P_1(D = H_0) \geq \min P_1(D = H_0) = \beta_{NP}(\alpha) \)
Estimation problem (Bayesian approach)

We are given a data vector $X = [x_1, \ldots, x_n]$ for which we assume the following model:

$$X \sim f_1(X|\theta), \quad \pi(\theta)$$

Use $X$ to find an estimate $\hat{\theta}$ of $\theta$

Minimize $E_1 \left[ ||\hat{\theta} - \theta||^2 \right]$ 

$$\hat{\theta}_o(X) = E_1[\theta|X] = \frac{\int \theta f_1(X|\theta)\pi(\theta)d\theta}{\int f_1(X|\theta)\pi(\theta)d\theta}$$
Formulation of the joint problem

\[ H_0 : \quad X \sim f_0(X) \]
\[ H_1 : \quad X \sim f_1(X|\theta), \quad \pi(\theta) \]

Ad-hoc

Optimum in each subproblem:

\[
\frac{\int f_1(X|\theta)\pi(\theta)d\theta}{f_0(X)} \quad \frac{H_1}{\nabla H_0} \quad t
\]

\[
\hat{\theta}_o = \frac{\int \theta f_1(X|\theta)\pi(\theta)d\theta}{\int f_1(X|\theta)\pi(\theta)d\theta}
\]

GLRT:

\[
\max_\theta \quad \frac{f_1(X|\theta)}{f_0(X)} \quad \frac{H_1}{\nabla H_0} \quad t
\]
One-step tests

We distinguish our decisions into $H_0$ and $H_1$ **BUT**

- $H_1$: detection **and** reliable estimation
- $H_0$: no detection **or** detection without reliable estimation

Minimize

$$E_1 \left[ \| \hat{\theta} - \theta \|^2 | D = H_1 \right]$$

We need to **control** the detection part

$$\alpha \geq P_0(D=H_1) \quad \beta \geq P_1(D=H_0) \geq \beta_{NP}(\alpha)$$
Theorem: The optimum combined detection and estimation scheme is defined as follows:

\[ \hat{\theta}_o(X) = \frac{\int \theta f_1(X|\theta)\pi(\theta)d\theta}{\int f_1(X|\theta)\pi(\theta)d\theta} \]

\[ \sigma^2(X) = E_1 \left[ ||\theta - \hat{\theta}_o||^2 | X \right] \]

\[ = \frac{\int ||\theta - \hat{\theta}_o||^2 f_1(X|\theta)\pi(\theta)d\theta}{\int f_1(X|\theta)\pi(\theta)d\theta} \]

\[ \frac{\int f_1(X|\theta)\pi(\theta)d\theta}{f_0(X)} \begin{cases} \lambda - \sigma^2(X) \end{cases} \begin{cases} H_1 \end{cases} \]

\[ t \begin{cases} H_0 \end{cases} \]
Detect if after some point, there is a change in the statistical behavior of the data.

Every time we detect a change, we like to estimate the point of change, i.e. the boundary.

Detect objects in images and find boundaries. Detection has no meaning without boundary estimation.
Two-step tests

We distinguish our decisions into \( H_0 \) and \( H_1 \). Every time I decide in favor of \( H_1 \) I compute an estimate for my parameters.

I can now ask myself: **Can I trust my estimate?** Is the estimate reliable (denoted as \( H_{1r} \)) or unreliable (denoted as \( H_{1u} \))?

Need a **second decision mechanism** that decides between \( H_{1r} \) and \( H_{1u} \).

We have three decisions: \( H_0, H_{1r} \) and \( H_{1u} \).
\[ \alpha \geq P_0(D=H_1) \quad \min \ P_1(D=H_0) \]

\[ \frac{\int f_1(X|\theta)\pi(\theta)d\theta}{f_0(X)} \]

\[ \min E_1[\|\hat{\theta} - \theta\|^2|D = H_{1r}] \]

\[ 1 - \beta \quad \frac{P(D = H_{1r})}{1 - \beta_{NP}(\alpha)} \leq \frac{P(D = H_1)}{P(D = H_{1r})} \]

\[ \hat{\theta}_o(X) = \frac{\int \theta f_1(X|\theta)\pi(\theta)d\theta}{\int f_1(X|\theta)\pi(\theta)d\theta} \]

\[ \sigma^2(X) \]

\[ \sigma^2(X) \]

\[ \sigma^2(X) \]

Decide \( H_{1r} / H_{1u} \)

MOUSTAKIDES: Optimum joint detection and estimation: Application to MIMO radar, USC, Jan. 2011
\[
\mathcal{L}(X|\theta) = \frac{f_1(X|\theta)}{f_0(X)}
\]

\[
\int \frac{f_1(X|\theta)\pi(\theta)d\theta}{f_0(X)} = \int \mathcal{L}(X|\theta)\pi(\theta)d\theta \quad \begin{cases} H_1 & t \\ H_0 \end{cases}
\]

\[
\hat{\theta}_o(X) = \frac{\int \theta \mathcal{L}(X|\theta)\pi(\theta)d\theta}{\int \mathcal{L}(X|\theta)\pi(\theta)d\theta}
\]

\[
\sigma^2(X) = \frac{\int \|\theta - \hat{\theta}_o\|^2 \mathcal{L}(X|\theta)\pi(\theta)d\theta}{\int \mathcal{L}(X|\theta)\pi(\theta)d\theta}
\]
MIMO radar

Measure the delay. The delay is proportional to the distance traveled by the EM wave. Defines an ellipse.
Consider a maximal allowable delay.

The operational space of the radar

Only detect

Detect & estimate
Application of optimum scheme
\[ r_i(t) = \sum_{j=1}^{K} g_{ij} \frac{s_j(t - \tau_{ij})}{d_{ij}^m} + w_i(t) \]

\[ g_{ij} \sim \mathcal{N}(0, 1); \quad w_i(t) \sim \mathcal{N}(0, 1) \]

Apply a LR test
\[ r_i(t) = \sum_{j=1}^{K} g_{ij} \frac{s_j(t - \tau_{ij})}{\tau_{ij}} + w_i(t) \]

\[ s_1(t), \quad s_2(t), \quad \ldots, \quad s_K(t) \]

\[ g_{ij} \sim \mathcal{N}(0, 1); \quad w_i(t) \sim \mathcal{N}(0, 1) \]

Treat the delays \( \tau_{ij} \) as unrelated and estimate them.

The \( KN \) delays generate \( KN \) ellipses and a set of at most \( KN \) choose 2 target positions. We average.

\[ r_1(t), \quad r_2(t), \quad \ldots, \quad r_M(t) \]
\[ r_i(t) = \sum_{j=1}^{K} g_{ij} \frac{s_j(t - \tau_{ij})}{\tau_{ij}} + w_i(t) \]

\[ g_{ij} \sim \mathcal{N}(0, 1); \quad w_i(t) \sim \mathcal{N}(0, 1) \]

Actually, \( \tau_{ij} = \tau_{ij}(x, y) \) are known functions of the target position.
\[ H_0 : r_i(t) = w_i(t) \]
\[ H_1 : r_i(t) = \sum_{j=1}^{K} g_{ij} \frac{s_j(t - \tau_{ij})}{\tau_{ij}} + w_i(t) \]
\[ g_{ij} \sim \mathcal{N}(0, 1); \quad w_i(t) \sim \mathcal{N}(0, 1) \]

For the position \((x, y)\) we assume uniform prior over the operational space of the radar.

\[ X \iff [r_1(t), \ldots, r_M(t)] \quad \theta \iff (x, y) \]
\[ \mathcal{L}(X | \theta) \iff \mathcal{L}(r_1(t), \ldots, r_M(t) | x, y) \]
\[
\mathcal{L}(r_1(t), \ldots, r_M(t) | x, y) = \prod_{i=1}^{M} \mathcal{L}(r_i(t) | x, y)
\]

\[
r_i(t) = \sum_{j=1}^{K} g_{ij} \frac{s_j(t - \tau_{ij})}{\tau_{ij}^\eta} + \omega_i(t)
\]

\[
= G_i' S(t - \mathcal{T}_i) + \omega_i(t)
\]

\[
G_i = [g_{i1}, \ldots, g_{iK}]', \quad \text{iid Gaussian } \mathcal{N}(0,1)
\]

\[
S(t - \mathcal{T}_i) = \left[ \frac{s_1(t - \tau_{i1})}{\tau_{i1}^\eta}, \ldots, \frac{s_K(t - \tau_{iK})}{\tau_{iK}^\eta} \right]
\]

\text{Known deterministic signals}

\text{Known deterministic functions of } (x, y)
Assume we measure $r_i(t)$ during the interval $[0,T]$ then

$$
\mathcal{L}(r_i(t)|x, y, G_i) = e^{-0.5 G'_i Q_i(x,y) G_i + G'_i R_i(x,y)}
$$

$$
Q_i(x,y) = \int_0^T S(t - T_i) S'(t - T_i) dt
$$

$$
R_i(x,y) = \int_0^T r^*_i(t) S(t - T_i) dt
$$

Integrating out $G_i$

$$
\mathcal{L}(r_i(t)|x, y) = e^{0.5 R'_i(x,y) \{Q_i(x,y) + I\}^{-1} R_i(x,y)} \frac{\det(Q_i(x,y) + I)}{
\det(Q_i(x,y) + I)}
$$
\[ \hat{\theta}_o = \frac{\int \theta \mathcal{L}(X | \theta) \pi(\theta) d\theta}{\int \mathcal{L}(X | \theta) \pi(\theta) d\theta} \]

Assuming **uniform** prior for the target position on the operational space of the radar and by sampling uniformly this space we have

\[ \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} \approx \sum_{(x_k, y_k) \in \text{op.sp.}} \left[ \begin{array}{c} x_k \\ y_k \end{array} \right] \frac{\prod_{i=1}^{M} \mathcal{L}(r_i | x_k, y_k)}{\sum_{(x_k, y_k) \in \text{op.sp.}} \prod_{i=1}^{M} \mathcal{L}(r_i | x_k, y_k)} \]
Simulations

\[ s_i(t) = E e^{j2\pi it/\Delta} \]

Duration \( \Delta = 10^{-4} \text{sec} \)
Integration period \( T = 5 \times 10^{-4} \text{sec} \)

Sampling: \( dx = dy = 10 \text{ Km} \)
generates 179 points

SNR = -20, -10, 0, 10 dB
False alarm: \( \alpha = 0.001 \)

Simulation is repeated 200,000 times
1) GLRT: 
\[
\max_{x,y} \prod_{i=1}^{M} \mathcal{L}(r_i \mid x, y) \quad H_1 \\
\leq \quad H_0 \\
\left[ \begin{array}{c} \hat{x} \\ \hat{y} \end{array} \right] = \arg \max_{x,y} \prod_{i=1}^{M} \mathcal{L}(r_i \mid x, y)
\]

2) For each subproblem use the optimum.

3) Proposed optimum two-step test.

We note that 2) is a special case of 3)