Sequential rate change detection in Poisson processes

George V. Moustakides
University of Patras, GREECE
Outline

- Overview of sequential changepoint detection
- CUSUM test and Lorden’s criterion
- The homogeneous Poisson disorder problem
  - CUSUM average run length
  - CUSUM optimality
- Non-homogeneous Poisson disorder problem and Epidemic surveillance
Overview

Using \( \{\xi_t\} \) detect \( \tau \) as soon as possible
Quality control
Systems monitoring
Remote sensing and GIS
Smart cameras – Human computer interaction
Image processing

Optical communications
Changepoint models for hazard functions
Occurrence of industrial accidents
Epidemic detection
Monitoring of link failures in computer networks
\( P_\infty \): nominal measure
\( P_0 \): alternative measure
\( P_\tau \): measure induced by the change
\( E_\tau [.] \): corresponding expectation

Both \( P_0 \) and \( P_\infty \) are assumed known
We are interested in **sequential schemes**

With every new data point $\xi_t$ we decide whether

- Stop and raise an alarm
- Continue sampling

For the decision at time $t$ we use all the available information up to time $t$

$$\mathcal{F}_t = \sigma\{\xi_s : 0 \leq s \leq t \}$$

Sequential test $\iff$ stopping time $T$ adapted to the filtration $\{\mathcal{F}_t\}$
Detection delay

\[ \mathcal{J}(T) = E_{\tau}[T - \tau | T > \tau] \]

what is \( \tau \)?

False alarm

Average period between false alarms:

\[ E_{\infty}[T] \geq \gamma \]

False alarm probability:

\[ P(T \leq \tau) \leq \alpha \]

\[ \min_{T} \mathcal{J}(T) \]
Changepoint mechanism

There is a mechanism that decides when to impose the change. This decision can be

- Independent from the observations $\{\xi_t\}$
- Depend on the observations $\{\xi_t\}$

If independent, then $\tau$ appears as random variable with prior

$$P(\tau = t) = \pi_t$$

If $\{\pi_t\}$ known then Bayesian formulation
Bayesian Formulation

Zero modified geometric

\[ \pi_t = (1 - \omega)p(1 - p)^{t-1}, \quad t \geq 1; \quad \pi_0 = \omega. \]

\[
\min_{T} E[T - \tau | T > \tau]; \quad \text{s.t.} \quad P(T \leq \tau) \leq \alpha
\]

\[ S_t = (S_{t-1} + 1) \frac{f_0(\xi_t)}{(1 - p)f_{\infty}(\xi_t)}, \quad S_0 = r \]

\[ T_{SR} = \inf \{ t : S_t \geq \nu \} \]

The mechanism decides without consulting the observations but \( P(\tau = t) = \pi_t \) is unknown.

\[
\mathcal{J}_P(T) = \sup_{\{\pi_t\}} \mathbb{E}_\tau [T - \tau | T > \tau]
\]

\[
= \sup_{t \geq 0} \mathbb{E}_t [T - t | T > t]
\]

Pollak’s (1985) criterion

\[
\inf_T \mathcal{J}_P(T) = \inf_T \sup_{t \geq 0} \mathbb{E}_t [T - t | T > t]
\]

s.t. \( \mathbb{E}_\infty [T] \geq \gamma \)
Pollak (1985) proposed the following stopping time known as Shiryaev-Roberts-Pollak

\[ S_t = (S_{t-1} + 1) \frac{f_0(\xi_t)}{f_\infty(\xi_t)}, \quad S_0 \sim q(S) \]

\[ T_{\text{SRP}} = \inf\{t : S_t \geq \nu\} \]

\[ E_t[T_{\text{SRP}} - t | T_{\text{SRP}} > t] = \text{constant} \]

\[ \mathcal{J}_P(T_{\text{SRP}}) - \inf_T \mathcal{J}_P(T) = o(1), \quad \text{as } \gamma \to \infty \]
Strict Optimality?

In 1997 appears a proof that the SRP test is optimum (Annals of Statistics).

Yajun Mei (2006), shows that the proof is problematic.

The conjecture remained unanswered until last year (2010): Tartakovsky and Polunchenko, produced a counterexample. After 25 years we can finally say that

The SRP test is NOT optimum.
CUSUM and Lorden’s criterion

The changepoint mechanism decides to impose the change by consulting the observations \( \{\xi_t\} \) and possibly additional information.

In this case \( \tau \) becomes a stopping time adapted to a larger filtration than \( \{\mathcal{F}_t\} \).

\[
\mathcal{J}_L(T) = \sup_{\tau} E_\tau[T - \tau | T > \tau]
\]

\[
= \sup_{t \geq 0} \text{esssup} E_t[T - t | T > t, \mathcal{F}_t]
\]

Lorden’s (1971) criterion
\[ X_t = AX_{t-1} + BW_t \]

\[ \xi_t = CX_t + Dw_t \]

Possible cause for change is for example “large” oscillations

\[ \tau = \inf \{ t : \|X_t\| \geq c \} \]

\[ A \rightarrow A' \]

A bomb set to explode at a specific time
\[ \inf_{T} \mathcal{I}_L(T) = \inf_{T} \sup_{t \geq 0} \mathbb{E}_t[T - t | T > t, \mathcal{F}_t] \]

\[ \mathbb{E}_\infty[T] \geq \gamma \]

The optimum scheme: CUSUM

\[ u_t = \log \left( \frac{d\mathbb{P}_0}{d\mathbb{P}_\infty}(\mathcal{F}_t) \right) \]

\[ m_t = \inf_{0 \leq s \leq t} u_s \]

\[ y_t = u_t - m_t \geq 0 \]

\[ T_C = \inf \{ t : y_t \geq \nu \} \]

I.i.d.: Moustakides (1986)
Ritov (1990), Poor (1998)
Ito: Moustakides (2004)
ML estimate of $\tau$
Homogeneous Poisson disorder

Let \( \{ \mathcal{N}_t \} \) denote a homogeneous Poisson process with rate \( \lambda \) satisfying

\[
\lambda = \begin{cases} 
\lambda_\infty & \text{for } t \leq \tau \\
\lambda_0 & \text{for } t > \tau.
\end{cases}
\]

\[
u_t = (\lambda_\infty - \lambda_0)t + \log \frac{\lambda_0}{\lambda_\infty} \mathcal{N}_t
\]

\[
u_t = a t + b \mathcal{N}_t
\]

\( a, b \) opposite signs

Show optimality of CUSUM

First step compute ARL: \( E_0[T_C] \) and \( E_\infty[T_C] \)
We are looking for \( f(y) = \mathbb{E}[T_C | y_0 = y] \)

\[
\mathbb{E}[f(y_{T_C})] - f(y_0) = \\
\mathbb{E} \left[ \int_0^{T_C} \left\{ af'(y_{t-}) + \lambda \left[ f(y_{t-} + b) - f(y_{t-}) \right] \right\} dt \right] = -1
\]
\[ af'(y) + \lambda [f(y + b) - f(y)] = -1 \]

\[ f(\nu) = 0; \quad f(y) = f(0) \quad y \leq 0 \]

\[ f(y) = \frac{1}{\lambda} \left\{ \sum_{n=0}^{[\frac{\nu}{|b|}]} \phi_n(\nu - n|b|) - \sum_{n=0}^{[\frac{y}{|b|}]} \phi_n(y - n|b|) \right\} \]

\[ \phi_n(y) = e^{\frac{\lambda y}{a}} \left( \sum_{k=0}^{n} \frac{(-\frac{\lambda y}{a})^k}{k!} \right) - 1; \]
Optimality of CUSUM

\[ J_L(T) = \sup_{t \geq 0} \text{essup} E_t [T - t | T > t, \mathcal{F}_t] \]

\[ E_\infty [T] \geq \gamma \]

\[ E_\infty [T] \geq E_\infty [T_C] = \gamma \quad \Rightarrow \quad J_L(T) \geq J_L(T_C) \]

\[ h(y) = E_\infty [T_C | y_0 = y] \]

\[ g(y) = E_0 [T_C | y_0 = y] \]

\[ \text{essup} E_t [T_C - t | T_C > t, \mathcal{F}_t] = g(0) \]

\[ J_L(T_C) = g(0) \quad \quad E_\infty [T_C] = h(0) \]
\[ E_\infty [T] \geq h(0) \Rightarrow J_L(T) \geq g(0) \]

**Lemma:**
\[ J_L(T) \geq \frac{E_\infty \left[ \int_0^T e^{y_t} \, dt \right]}{E_\infty [e^{y_T}]} \]

\[ E_\infty [T] \geq h(0) \Rightarrow \frac{E_\infty \left[ \int_0^T e^{y_t} \, dt \right]}{E_\infty [e^{y_T}]} \geq g(0) \]

\[ E_\infty \left[ \int_0^T e^{y_t} \, dt \right] = g(0) E_\infty [e^{y_T}] + E_\infty [T] - h(0) \geq 0 \]

\[ + E_\infty [h(y_T) - e^{y_T} g(y_T)] \geq 0 \]

**Lemma:**
\[ h(y) - e^y g(y) \geq 0, \quad \forall \ y \geq 0 \]
Nonhomogeneous Poisson

Let $\{N_t\}$ denote a nonhomogeneous Poisson process with rate $\lambda_t$ that satisfies

$$
\lambda_t = \begin{cases} 
\omega_t & \text{for } t \leq \tau \\
\rho \omega_t & \text{for } t > \tau,
\end{cases}
$$

$\omega_t$ is adapted to our observations (that can include more information than $\{N_t\}$) and $\rho$ is a known constant.
CUSUM test:

\[ u_t = (1 - \rho) \int_0^t \omega_s \, ds + (\log \rho) N_t \]

\[ m_t = \inf_{0 \leq s \leq t} u_s \]

\[ y_t = u_t - m_t \]

\[ T_C = \inf \{ t : y_t \geq \nu \} \]

Is it optimum? **YES!** but in a Lorden-like sense

\[ J_L(T) = \sup_{t \geq 0} \text{esssup} E_t [N_T - N_t | T > t, F_t] \]

\[ E_\infty [N_T] \geq \gamma \]
Epidemic Detection

Detect the onset of an epidemic: when the incidence rate of a particular disease increases significantly above some standard level

\( \omega_t \) : describes the nominal incidence rate, which can be a function of several observable quantities as population, pollution measurements etc.

\( \rho > 1 \) : smallest increase in incidence rate that should be considered ... alarming.

Detect rapidly in number of incidences