Optimum CUSUM Tests
for
Detecting Changes in
Continuous Time Processes

George V. Moustakides
INRIA, Rennes, France
Outline

- The change detection problem
- Overview of existing results
- Lorden’s criterion and the CUSUM test
- A modified Lorden criterion
- CUSUM tests for Ito processes
- Extensions
The change detection problem

We are observing sequentially a process \( \{ \xi_t \} \) with the following statistics:

\[
\begin{align*}
\xi_t & \sim P_\infty \quad \text{for } 0 \leq t \leq \tau \\
& \sim P_0 \quad \text{for } \tau < t
\end{align*}
\]

**Goal:** Detect the change time \( \tau \) “as soon as possible”

- Change time \( \tau \): deterministic (but unknown) or random
- Probability measures \( P_\infty, P_0 \): known

Applications include: systems monitoring; quality control; financial decision making; remote sensing (radar, sonar, seismology); speech/image/video segmentation; …
The observation process \( \{ \xi_t \} \) is available sequentially. This can be expressed through the filtration:

\[
\mathcal{F}_t = \sigma\{ \xi_s : 0 < s \leq t \}.
\]

- Interested in **sequential detection schemes**. At every time instant \( t \) we perform a test to decide whether to stop and declare an alarm or continue sampling. The test at time \( t \) must be based on the available information up to time \( t \).

- Any sequential detection scheme can be represented by a **stopping time** \( T \) adapted to the filtration \( \mathcal{F}_t \) (the time we stop and declare an alarm).
Overview of existing results

$P_\tau$: the probability measure induced, when the change takes place at time $\tau$

$E_{\tau}[.]$: the corresponding expectation

$P_\infty$: all data under nominal regime

$P_0$: all data under alternative regime

Optimality criteria

They must take into account two quantities:
- The detection delay $T - \tau$
- The frequency of false alarms

Possible approaches: **Baysian and Min-max**
**Baysian approach** (Shiryayev 1978)

The change time $\tau$ is random with exponential prior.

For any stopping time $T$ define the criterion:

$$J(T) = c \mathbb{E}[ (T - \tau)^+ ] + \mathbb{P}[ T < \tau ]$$

**Optimization problem:** $\inf_T J(T)$

Compute the statistics: $\pi_t = \mathbb{P}[ \tau \leq t \mid \mathcal{F}_t ]$;

and stop: $T_S = \inf_t \{ t : \pi_t \geq \nu \}$

- **Discrete time:** when $\{\xi_n\}$ is i.i.d. and there is a change in the pdf from $f_{\infty}(\xi)$ to $f_0(\xi)$.

- **Continuous time:** when $\{\xi_t\}$ is a Brownian Motion and there is a change in the constant drift from $\mu_{\infty}$ to $\mu_0$. 

**Min-max approach** (Shiryayev-Roberts-Pollak)

The change time $\tau$ is deterministic but unknown.

For any stopping time $T$ define the criterion:

$$J(T) = \sup_{\tau} \mathbb{E}_{\tau}[ (T - \tau)^+ | T > \tau]$$

**Optimization problem:**

$$\inf_T J(T);$$

subject to:

$$\mathbb{E}_\infty[ T ] \geq \gamma$$

**Discrete time:** when $\{\xi_n\}$ is i.i.d. and there is a change in the pdf from $f_\infty(\xi)$ to $f_0(\xi)$.

Compute the statistics:

$$S'_n = (S'_{n-1} + 1) \frac{f_0(\xi_n)}{f_\infty(\xi_n)}.$$

and stop (Yakir 1997): $$T_{SRP} = \inf_n \{ n: S'_n \geq \nu \}$$
Lorden’s criterion and the CUSUM test

Alternative min-max approach (Lorden 1971):

The change time $\tau$ is deterministic and unknown. For any stopping time $T$ define the criterion:

$$J(T) = \sup_{\tau} \text{essup} \ E_\tau[(T - \tau)^+ | F_\tau]$$

Optimization problem: $\inf_T J(T)$;
subject to: $E_\infty[T] \geq \gamma$.

The test closely related to Lorden’s criterion and being the most popular test for the change detection problem in practice, is the **Cumulative Sum (CUSUM)** test.
Define the CUSUM process $y_t$ as follows:

$$y_t = u_t - m_t$$

where

$$u_t = \log\left( \frac{d\mathbb{P}_0}{d\mathbb{P}_\infty} (\mathcal{F}_t) \right)$$

$$m_t = \inf_{0 \leq s \leq t} u_s.$$ 

The CUSUM stopping time (Page 1954):

$$T_C = \inf_{t} \left\{ t : y_t \geq \nu \right\}$$

- **Discrete time**: when $\{\xi_n\}$ is i.i.d. before and after the change (Moustakides 1986, Ritov 1990).

- **Continuous time**: when $\{\xi_t\}$ is a Brownian Motion with constant drift before and after the change (Shiryayev 1996, Beibel 1996).
A modified Lorden criterion

We intend to extend the optimality of CUSUM to detection of changes in Ito processes by modifying Lorden’s criterion using the Kullback-Leibler Divergence (KLD).

Similar extension exists for the Sequential Probability Ratio Test (SPRT), applied in hypotheses testing, since 1978 (Liptser and Shiryayev)

The observation process \( \{\xi_t\} \) satisfies the following sde:

\[
d\xi_t = \begin{cases} 
  dw_t & 0 \leq t \leq \tau \\
  \alpha_t \, dt + dw_t & \tau < t 
\end{cases}
\]

\( \{w_t\} \) standard Brownian Motion

\( \{\alpha_t\} \) adapted to the history \( \mathcal{F}_t = \sigma\{\xi_s : 0 \leq s \leq t\} \)

If \( \alpha_t = \alpha(\xi_t) \), then \( \xi_t \) is a diffusion process for \( t > \tau \).
To \( \{\xi_t\} \) we correspond the following process \( \{u_t\} \)

\[
d u_t = \alpha_t \, d\xi_t - 0.5 \alpha_t^2 \, dt.
\]

We would like: \( u_t = \log\left( \frac{d\mathbb{P}_0}{d\mathbb{P}_\infty}(\mathcal{F}_t) \right) \).

We need the following conditions:

1. \( \mathbb{P}_0\left[ \int_0^t \alpha_s^2 \, ds < \infty \right] = \mathbb{P}_\infty\left[ \int_0^t \alpha_s^2 \, ds < \infty \right] = 1 \)

2. A “Novikov” condition

3. \( \mathbb{P}_0\left[ \int_0^\infty \alpha_s^2 \, ds = \infty \right] = \mathbb{P}_\infty\left[ \int_0^\infty \alpha_s^2 \, ds = \infty \right] = 1 \)
From Condition 1&2 we have validity of Girsanov’s theorem:

\[
\frac{d\mathbb{P}_0}{d\mathbb{P}_\infty}(\mathcal{F}_t) = e^{u_t} \quad \frac{d\mathbb{P}_\tau}{d\mathbb{P}_\infty}(\mathcal{F}_t) = e^{u_t-u_\tau}
\]

The Kullback-Leibler Divergence can then be written as:

\[
\mathbb{E}_\tau\left[ \log\left( \frac{d\mathbb{P}_\tau}{d\mathbb{P}_\infty}(\mathcal{F}_t) \right) \Big| \mathcal{F}_\tau \right] = \mathbb{E}_\tau\left[ \int_\tau^t \alpha_s \, dw_s + 0.5 \int_\tau^t \alpha_s^2 \, ds \Big| \mathcal{F}_\tau \right] = \mathbb{E}_\tau\left[ 0.5 \int_\tau^t \alpha_s^2 \, ds \Big| \mathcal{F}_\tau \right], \quad 0 \leq \tau \leq t
\]
The original Lorden criterion

\[ J(T) = \sup_{\tau} \text{essup} \ E_{\tau}[ (T - \tau)^+ | \mathcal{F}_{\tau} ] \]

using the Kullback-Leibler Divergence can be modified as

\[ J(T) = \sup_{\tau} \text{essup} \ E_{\tau} \left[ 0.5 \int_{\tau}^{T} \alpha_t^2 \, dt \right] \]

The two criteria are equivalent in the case

\[ \alpha_t^2 = \text{constant} \]

i.e. Brownian motion with constant drift.
Similarly

\[ \mathbb{E}_\infty \left[ \log \left( \frac{d\mathbb{P}_\infty}{d\mathbb{P}_0} (\mathcal{F}_t) \right) \right] \]

\[ = \mathbb{E}_\infty \left[ - \int_0^t \alpha_s \, dw_s + 0.5 \int_0^t \alpha_s^2 \, ds \right] \]

\[ = \mathbb{E}_\infty \left[ 0.5 \int_0^t \alpha_s^2 \, ds \right] \]

This suggest replacing the constraint \( \mathbb{E}_\infty [T] \geq \gamma \) with

\[ \mathbb{E}_\infty \left[ 0.5 \int_0^T \alpha_t^2 \, dt \right] \geq \gamma \]
Summarizing:

$$J(T) = \sup_\tau \text{essup } \mathbb{E}_\tau \left[ \mathbb{1}_{\{T > \tau\}} 0.5 \int_\tau^T \alpha_t^2 \, dt \mid \mathcal{F}_\tau \right]$$

**Optimization problem:**

$$\inf_T J(T);$$

subject to: $$\mathbb{E}_\infty \left[ 0.5 \int_0^T \alpha_t^2 \, dt \right] \geq \gamma$$
CUSUM tests for Ito processes

The CUSUM statistics $y_t$ for Ito processes takes the form

$$
du_t = \alpha_t \, d\xi_t - 0.5 \alpha_t^2 \, dt
$$

$$
m_t = \inf_{0 \leq s \leq t} u_s
$$

$$
y_t = u_t - m_t
$$

and the optimum CUSUM test is

$$
T_C = \inf_t \{ t : y_t \geq \nu \}
$$

where $\nu$ such that:

$$
E_\infty \left[ 0.5 \int_0^{T_C} \alpha_t^2 \, dt \right] = \gamma
$$

Since $y_t$ has continuous paths, when the CUSUM test stops we have: $y_{T_C} = \nu$. 
Since $u_t \geq m_t$ we conclude $y_t = u_t - m_t \geq 0$

$m_t$ is nonincreasing and $dm_t \neq 0$ only when $u_t = m_t$ or $y_t = u_t - m_t = 0$

If $f(y)$ continuous with $f(0) = 0$, then $f(y_t)dm_t = 0$
If \( f(y) \) is a twice continuously differentiable function with \( f'(0) = 0 \), using standard Ito calculus, we can write

\[
df(y_t) = f'(y_t)(du_t - dm_t) + 0.5\alpha_t^2f''(y_t)dt
\]

\[
= f'(y_t)du_t + 0.5\alpha_t^2f''(y_t)dt
\]

**Theorem 1:** \( T_C \) is a.s. finite, furthermore

\[
\mathbb{E}_\tau \left[ \mathbf{1}_{\{T_C > \tau\}} 0.5 \int_\tau^{T_C} \alpha_t^2 dt \mid \mathcal{F}_\tau \right] = [g(\nu) - g(y_\tau)] \mathbf{1}_{\{T_C > \tau\}}
\]

\[
\mathbb{E}_\infty \left[ \mathbf{1}_{\{T_C > \tau\}} 0.5 \int_\tau^{T_C} \alpha_t^2 dt \mid \mathcal{F}_\tau \right] = [h(\nu) - h(y_\tau)] \mathbf{1}_{\{T_C > \tau\}}
\]

\[
g(y) = y + e^{-y} - 1 \quad \quad h(y) = e^y - y - 1
\]
The functions $g(y), h(y)$ are increasing, strictly convex, with $g(0) = h(0) = 0$. We can therefore conclude

$$J(T_C) = \sup_\tau \text{essup} \mathbb{E}_\tau \left[ 1_{\{T_C > \tau\}} 0.5\int_\tau^{T_C} \alpha_t^2 \, dt \mid \mathcal{F}_\tau \right]$$

$$= \sup_\tau \text{essup}[g(\nu) - g(y_\tau)] 1_{\{T_C > \tau\}}$$

$$= g(\nu) - g(0) = g(\nu) = \nu + e^{-\nu} - 1$$

Similarly

$$\mathbb{E}_\infty \left[ 0.5 \int_0^{T_C} \alpha_t^2 \, dt \right] = h(\nu) - h(0) = h(\nu) = \gamma$$

$$e^\nu - \nu - 1 = \gamma$$
For any stopping time $T$, using again standard Itô calculus, we have the following corollary of Theorem 1

**Corollary:**

$$
\mathbb{E}_\tau \left[ \mathbbm{1}_{\{T > \tau\}} 0.5 \int_{\tau}^{T} \alpha_t^2 \, dt \right] = \mathbb{E}_\tau [g(y_T) - g(y_\tau) | \mathcal{F}_\tau] \mathbbm{1}_{\{T > \tau\}}
$$

$$
\mathbb{E}_\infty \left[ \mathbbm{1}_{\{T > \tau\}} 0.5 \int_{\tau}^{T} \alpha_t^2 \, dt \right] = \mathbb{E}_\infty [h(y_T) - h(y_\tau) | \mathcal{F}_\tau] \mathbbm{1}_{\{T > \tau\}}
$$

**Remark 1:** The false alarm constraint can be written as

$$
\mathbb{E}_\infty \left[ 0.5 \int_{0}^{T} \alpha_t^2 \, dt \right] = \mathbb{E}_\infty [h(y_T) - h(0)] = \mathbb{E}_\infty [h(y_T)] \geq \gamma
$$
**Remark 2:** The modified performance measure $J(T)$ can be suitably lower bounded as follows

$$J(T) = \sup_\tau \operatorname{essup} E_\tau \left[ \mathbf{1}_{\{T > \tau\}} \cdot 0.5 \int_\tau^T \alpha_t^2 \, dt \mid F_\tau \right]$$

$$\geq \frac{E_\infty[e^{y_T}g(y_T)]}{E_\infty[e^{y_T}]}$$

In the case of CUSUM the lower bound coincides with the corresponding performance measure $J(T_C)$.

**Remark 3:** We can limit ourselves to stopping times that satisfy the false alarm constraint with equality, i.e.

$$E_\infty[h(y_T)] = \gamma = h(\nu)$$
Theorem 2: Any stopping $T$ that satisfies the false alarm constraint with equality has a performance measure $J(T)$ that is no less than $J(T_C) = g(\nu)$.

Proof: Let $T$ satisfy the false alarm constraint with equality, i.e.

$$\mathbb{E}_\infty[h(y_T)] = \gamma = h(\nu)$$

we then like to show that: $J(T) \geq g(\nu)$.

Since $J(T) \geq \frac{\mathbb{E}_\infty[e^{y_T}g(y_T)]}{\mathbb{E}_\infty[e^{y_T}]}$ it is sufficient to show

$$\mathbb{E}_\infty[e^{y_T}\{g(y_T) - g(\nu)\} + h(\nu) - h(y_T)] \geq 0$$
If we define the function

\[ p(y) = e^y \{ g(y) - g(\nu) \} + h(\nu) - h(y) \]

then the previous inequality becomes: \( \mathbb{E}_\infty [p(y_T)] \geq 0 \)

We observe that \( p(y) \geq 0 \)

therefore we also have \( \mathbb{E}_\infty [p(y_T)] \geq 0 \)

**with equality iff** \( y_T = \nu \), i.e. the CUSUM test.
Extensions

Can our result be extended to the discrete time case?

\[ \xi_n = \begin{cases} 
  w_n & 0 \leq n \leq \tau \\
  \alpha_{n-1} + w_n & \tau < n 
\end{cases} \]

\{w_n\} an i.i.d. Gaussian process
\{\alpha_n\} adapted to the history \( \mathcal{F}_n = \sigma\{\xi_k : 0 \leq k \leq n\} \)

Not Straightforward!

\[ \mathbb{E} \left[ \mathbbm{1}_{\{T > \tau\}} \cdot 0.5 \sum_{k=\tau}^{T} \alpha_k^2 \bigg| \mathcal{F}_\tau \right] \neq \mathbb{E} \left[ g(y_T) - g(y_\tau) \bigg| \mathcal{F}_\tau \right] \mathbbm{1}_{\{T > \tau\}} \]

Similar problem exists for SPRT.
Straightforward extension for scalar processes

\[ d\xi_t = \begin{cases} 
\alpha_t \, dt + \sigma_t \, dw_t & 0 \leq t \leq \tau \\
\beta_t \, dt + \sigma_t \, dw_t & \tau < t 
\end{cases} \]

or vector processes

\[ d\Xi_t = \begin{cases} 
A_t \, dt + \sum_t dW_t & 0 \leq t \leq \tau \\
B_t \, dt + \sum_t dW_t & \tau < t 
\end{cases} \]
EnD