Problem 1: A pair of random variables \((y_1, y_2)\) can take upon four different values: \((2, 0)\) with probability 0.08, \((0, 2)\) with probability 0.12, \((0, -2)\) with probability 0.32, and \((-2, 0)\) with probability 0.48. Using Independent Component Analysis (ICA) find the linear transformation \(x_1 = a_{11} y_1 + a_{12} y_2\), \(x_2 = a_{21} y_1 + a_{22} y_2\) that transforms \((y_1, y_2)\) into independent variables \((x_1, x_2)\). Write the necessary equation without solving the resulting system that can compute the coefficients \(a_{ij}\). Explain why this system is sufficient to obtain a solution. Explain how the transformation obtained by solving the system you propose can help you recover the independent components. 

*Hint: ICA was introduced and detailed in Lecture 7. You first require the covariance matrix of the transformed variables to be equal to the identity and then you need to come up with one more relationship (equation) in order to be able to find a solution. The system of equations you will end up with must contain only the parameters \(a_{ij}\) of the transformation.*

Problem 2: In the figure appears a graph that we like to (page)rank its nodes. If you are at a node then 1) with probability \(\alpha = 0.8\) you select with equal probability to follow one of the outgoing edges and arrive at the next node. 2) with probability \(1 - \alpha = 0.2\) you are tele-transported to any node. During tele-transportation all nodes are eligible with the same probability to be visited (including the node currently visited). Provide the equations that compute the steady-state probabilities of the nodes and use these values to rank the nodes according to their visiting frequency. 

*Hint: PageRank was taught in Lecture 8.*

Problem 3: Consider the following utility matrix in a recommendation system:

\[
M = \begin{bmatrix}
2 & 1 & 1 & 5 \\
4 & 2 & 3 & 5 \\
3 & 3 & 5 & 1 \\
5 & 1 & 2
\end{bmatrix}
\]

where vertically we have users and horizontally movies. The values in each cell denotes evaluation (0-5) of the movie by the corresponding user. We would like to apply a decomposition to fill the utility matrix in order to make recommendations. Assuming movies are labeled 1 through 6 and users \(A\) through \(D\), we would like to apply a decomposition to the utility matrix \(M\) of the form \(M = UV\) where \(U\) has dimensions \(4 \times d\) and \(V\) has \(d \times 6\). If we follow the method we detailed in class where we switch from fixing \(U\) and optimizing \(V\) to fixing \(V\) and optimizing \(U\), what is the largest dimension \(d\) we can use? For this largest dimension \(d\) iterate between the two subproblems and compute a possible \(\hat{M}\). From the numerical results you have obtained what is your recommendation to each user regarding the movies he or she has not seen? 

*Hint: Recommendation systems were taught in Lecture 9.*

Your reports, in hard copy, must be submitted to me tomorrow, Tuesday, November 6, between 5:00-6:00PM, Room: CBIM-03. (NOT SOONER and NOT LATER!!!)

There will be no meeting in Hill 101 and you are not allowed to ask me for any help.