Section 1.6
Sets

A set is a collection or group of objects or elements or members. (Cantor 1895)

- A set is said to contain its elements.
- There must be an underlying universal set U, either specifically stated or understood.

Notation:

- list the elements between braces:

\[ S = \{ a, b, c, d \} = \{ b, c, a, d, d \} \]

(Note: listing an object more than once does not change the set. Ordering means nothing.)

- specification by predicates:

\[ S = \{ x \mid P(x) \}, \]

S contains all the elements from U which make the predicate \( P \) true.

- brace notation with ellipses:

\[ S = \{ \ldots, -3, -2, -1 \}, \]

the negative integers.
Common Universal Sets

- $R$ = reals
- $N$ = natural numbers = \{0, 1, 2, 3, \ldots \}, the counting numbers
- $Z$ = all integers = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots \}
- $Z^+$ is the set of positive integers

Notation:

$x$ is a member of $S$ or $x$ is an element of $S$:

$x \in S.$

$x$ is not an element of $S$:

$x \notin S.$
SET THEORY

A set is a collection of objects - members:

$$A = \{a, b, c\}$$

All these are the same:

$$\{a, b, c\} \quad \{b, c, a\} \quad \{b, b, a, c, a\}$$

order is irrelevant

Ex:

$$V = \{a, e, i, o, u\}$$

$$\emptyset = \{1, 3, 5, 7, 9\}$$

$$X = \{a, 2, Fred, New Jersey\}$$

$$I = \{0, 1, 2, 3, \ldots\}$$
$a \in S$
$a$ is an element of $S$
$a$ is a member of $S$
$a$ is in $S'$
$a$ belongs to $S'$

$a \notin S$

$1 \in \{3, 1, 2\}$
$u \in \{v, u, x, \omega\}$
$r \notin \{a, b, c\}$
principle of specification (intension)

\[ A = \{ x | P(x) \} \]

the set of all \[ x \text{ such that} P(x) \]

\[ O = \{ 1, 3, 5, 7, 9 \} \]

\( O = \{ x \mid x \text{ is an odd positive integer less than 10} \} \)

\[ R = \{ x \mid x \text{ is a real number} \} \]

\[ R : \text{real numbers} \]

\[ \mathbb{Z} : \text{set of all integers (Zahlen)} \]

\[ \mathbb{Q} : \text{rational numbers} \]

\[ \mathbb{Z}^+ \subseteq \mathbb{Z} \text{ non neg.} \]
\[ \{ x \in \mathbb{R} \mid -2 < x < 5 \} \]

\[ \{ x \in \mathbb{Z} \mid -2 < x < 5 \} \]

\[ = \{ -1, 0, 1, 2, 3, 4 \} \]

\[ \{ x \in \mathbb{Z}^+ \mid -2 < x < 5 \} \]

\[ = \{ 1, 2, 3, 4 \} \]

...
Venn Diagrams

A useful geometric visualization tool (for 3 or less sets)

- The Universe U is the rectangular box
- Each set is represented by a circle and its interior
- All possible combinations of the sets must be represented

For 2 sets

For 3 sets

Shade the appropriate region to represent the given set operation.
Two sets are equal if and only if they have the same elements.

\[ A = \{ a, b, c \} \]
\[ B = \{ b, c, a \} \]
\[ A = B \]

Any element in \( A \) is also an element in \( B \).

Any element in \( B \) is also an element in \( A \).
A is a subset of B: every element of A is also an element of B.
Proper subset

A is a proper subset of B

\[ B \supseteq A \]

every element in A is also an element in B and
there is at least one element in B that is not in A.

Note
Every set is a subset of itself

\[ A \subseteq A \]
Empty Set $\emptyset$

A set that doesn't contain any members.

$\emptyset = \{ \}$

Also called: null set - void set

$\emptyset$ is a subset of every set

why?

All the elements in $\emptyset$ are also elements in any other set!!
True or False

\[ x \in \{ x \} \]
\[ \{ x \} \subseteq \{ x \} \]
\[ \{ x \} \in \{ \{ x \} \} \]
\[ \emptyset \subseteq \{ x \} \]
\[ \emptyset \in \{ x \} \]
Power Set

The set of all subsets of a set $A$ is called the power set of $A$, denoted $P(A)$.

$A = \{a, b\}$

what are the possible subsets of $A$

$\{a\}$

$\{b\}$

$\{a, b\}$

$\emptyset$

$P(A) = \{ \{a\}, \{b\}, \{a, b\}, \emptyset \}$

Note: Power set is a set of sets
**Definition:** The number of (distinct) elements in A, denoted \(|A|\), is called the *cardinality* of A.

If the cardinality is a natural number (in \(N\)), then the set is called *finite*, else *infinite*.

Example:

\[ A = \{a, b\}, \]

\[ |\{a, b\}| = 2, \]

\[ |P(\{a, b\})| = 4. \]

A is finite and so is \(P(A)\).

Useful Fact: \(|A|=n\) implies \(|P(A)| = 2^n\)

N is infinite since \(|N|\) is not a natural number. It is called a *transfinite cardinal number*.

Note: Sets can be both *members* and *subsets* of other sets.
Set Union

The union of the sets $A$ and $B$, denoted by $A \cup B$, is the set of all elements that are either in $A$ or in $B$ or in both.

Ex: $A = \{a, b, c, y\}$
    $B = \{b, c, d, e, y\}$

$A \cup B = \{a, b, c, d, e, y\}$

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$B$</td>
</tr>
<tr>
<td>[\text{mark } A \cup B]</td>
<td></td>
</tr>
</tbody>
</table>
Set Intersection

The intersection of the sets $A$ and $B$, denoted by $A \cap B$, is the set containing those elements in both $A$ and $B$ (common elements).

Example:

$A = \{a, b, c\}$

$B = \{b, c, d, e\}$

$A \cap B = \{b, c\}$
Disjoint Sets

Two sets are called disjoint if their intersection is the empty set (have no common elements)

Ex: \[ A = \{a, b, c\} \]
\[ B = \{d, e, f\} \]
\[ A \cap B = \{ \} = \emptyset \]

A and B are disjoint sets

[Diagram showing sets A and B with no overlap]
Set Difference

the difference of $A$ and $B$, denoted by $A - B$, is the set containing elements that are in $A$ but not in $B$

Similarly $B - A$, set containing elements in $A$ but not in $B$

Ex: $A = \{a, b, c\}$
$B = \{b, c, d, e\}$

$A - B = \{a\}$
$B - A = \{d, e\}$

$A - B$ is also called complement of $B$ w.r.t. $A$
Complement

The complement of $A$, denoted by $\overline{A}$, is the complement of $A$ w.r.t. $U$

i.e. $\overline{A} = U - A$

(these elements that are not in $A$)

Ex: $A = \{a, e, i, o, u\}$

what is $\overline{A}$ (where the universal set is the English alphabet)

$\overline{A} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$
The Cartesian Product of $A$ and $B$, denoted by $A \times B$, is the set of all ordered pairs $(a,b)$ where $a \in A$, $b \in B$.

Example:

$A = \{a, b\}$

$B = \{b, c, d\}$

$A \times B = \{(a, b), (a, c), (a, d), (b, b), (b, c), (b, d)\}$
Examples: \( U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \)
\( A = \{1, 2, 3, 4, 5\}, B = \{4, 5, 6, 7, 8\} \). Then

- \( A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\} \)
- \( A \cap B = \{4, 5\} \)
- \( \overline{A} = \{0, 6, 7, 8, 9, 10\} \)
- \( \overline{B} = \{0, 1, 2, 3, 9, 10\} \)
- \( A - B = \{1, 2, 3\} \)
- \( B - A = \{6, 7, 8\} \)
- \( A \oplus B = \{1, 2, 3, 6, 7, 8\} \)
<table>
<thead>
<tr>
<th>Identity</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \cup \emptyset = A$</td>
<td>Identity laws</td>
</tr>
<tr>
<td>$A \cap U = A$</td>
<td></td>
</tr>
<tr>
<td>$A \cup U = U$</td>
<td>Domination laws</td>
</tr>
<tr>
<td>$A \cap \emptyset = \emptyset$</td>
<td></td>
</tr>
<tr>
<td>$A \cup A = A$</td>
<td>Idempotent laws</td>
</tr>
<tr>
<td>$A \cap A = A$</td>
<td></td>
</tr>
<tr>
<td>$\overline{A} = A$</td>
<td>Complementation law</td>
</tr>
<tr>
<td>$A \cup B = B \cup A$</td>
<td>Commutative laws</td>
</tr>
<tr>
<td>$A \cap B = B \cap A$</td>
<td></td>
</tr>
<tr>
<td>$A \cup (B \cup C) = (A \cup B) \cup C$</td>
<td>Associative laws</td>
</tr>
<tr>
<td>$A \cap (B \cap C) = (A \cap B) \cap C$</td>
<td></td>
</tr>
<tr>
<td>$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$</td>
<td></td>
</tr>
<tr>
<td>$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$</td>
<td></td>
</tr>
<tr>
<td>$\overline{A \cup B} = \overline{A} \cap \overline{B}$</td>
<td>De Morgan’s laws</td>
</tr>
<tr>
<td>$\overline{A \cap B} = \overline{A} \cup \overline{B}$</td>
<td></td>
</tr>
</tbody>
</table>
What can you say about the sets $A, B$ if the following is true

$A \cup B = A$

$A \cap B = A$

$A \setminus B = A$

$\overline{A \cap B} = A \cup B$ disjoint

$A \cap B = \emptyset$

$A = B$

$A \subseteq B$