Implication – Conditional Statement

p → q (p implies q) (if p then q) is the proposition that is false when p is true and q is false and true otherwise.

Equivalent to "not p or q"

Ex.

If I am elected then I will lower the taxes

If you get 100% on the final then you will get an A

p: I am elected

q: I will lower the taxes

Think of it as a contract, obligation or pledge

The contract is breached (implication is false) only if the p is true and q is false. Otherwise the contract (the pledge) is valid (true)

p → q (p implies q)

p: hypothesis, premise, antecedent

q: conclusion, consequence

p and q do not have to be related

If 2+2 = 4 then Albany is the capitol of N.Y. (T)

If 2+2 = 4 then Peapack is the capitol of N.J. (F)

If 2+2 = 5 then there is a state with only one neighbor
   (T)

If 2+2 = 5 then you are the pope

Also (T)

- Different forms for implications

p implies q   is equivalent to

If p then q
q if p
q whenever p
q when p
q follows from p

p is sufficient for q
a sufficient condition for q is p
q is necessary for p
a necessary condition for p is q

p only if q

$p \rightarrow q$ (p implies q)

p is sufficient for q

a sufficient condition for q is p

q is necessary for p

a necessary condition for p is q

Ex.

If you get 100% on the final you will get an A

You get 100% on the final is sufficient to get an A

A sufficient condition to get an A is to get 100% on the final

You get an A is necessary for you to get 100% on the final (but not sufficient)

A necessary condition for you get 100% is you get an A

(imagine that you know your letter grade and you are trying to guess your grade in the final:

If you didn't get A then for sure you didn't get 100%

If you get A then you may or may not get 100%)

p → q (p implies q)

p is sufficient for q

a sufficient condition for q is p

q is necessary for p

a necessary condition for p is q

If you show up on Monday you will get the job

You show up on Monday is sufficient for you to get the job

A sufficient condition for you to get the job is to show up on Monday

You get the job is necessary for you have shown up on Monday.

A necessary condition for you have shown up on Monday is you got the job.

- You walk 8 miles is necessary to get to the top

Which is equivalent ?

If you walk 8 miles then you get to the top

If you got to the top then you have walked 8 miles

The first statement is not equivalent. Walking 8 miles is necessary (but other things might be also necessary). suppose you walked 8 miles in the wrong direction ! But if you got to the top then you are sure that you must have walked 8 miles.

- Contrapositive

The contrapositive of "if p then q" is
"if not q then not p"

The contrapositive of
$p \rightarrow q$ is $\neg q \rightarrow \neg p$

A conditional statement is logically
   equivalent to its contrapositive.

Proof by truth table.

Ex. If today is Easter then tomorrow is
   Monday

Contrapositive:

If tomorrow is not Monday then today is not
   Easter

- If p then q   (p → q)

- The converse is " If q then p"

    q → p

- The inverse is "If not p then not q"

    ¬ p → ¬ q

- The contrapositive "if not q then not p"

    ¬ q → ¬ p

Ex. If today is Easter then tomorrow is Monday

Converse: If tomorrow is Monday then today is Easter

Inverse: If today is not Easter then tomorrow is not Monday

Conditional statement is not equiv. to its converse

Conditional statement is not equiv. to its inverse

The converse and inverse are logically equivalent.

- p only if q means "if not q then not p"

By contrapositive this is equivalent to "if p then q"

Ex:

If you get 100% on the final then you will get an A

Which statement is equivalent ?

- you will get A only if you get 100% on the final
- You get 100% on the final only if you get an A

The first statement is not equivalent to our statement: according to our statement you can get A even if you don't get 100% on the final.

- Your guarantee is good only if you bought your CD player less than 90 days ago

Which statement is equivalent ?

1. if you bought your CD player less than 90 days ago then your guarantee is good
2. If your guarantee is good then you have bought your CD player less than 90 days ago
3. You bought your CD player less than 90 days ago is sufficient for the guarantee to be good.
4. You bought your CD player less than 90 days ago is necessary for the guarantee to be good.

1 and 3 are not equivalent .. Why ?

The *biconditional* $\ p \leftrightarrow q\ $ is the proposition that is true when p and q have the same truth values, and is false otherwise.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional is equivalent to two way implication:

$$p \leftrightarrow q \equiv p \rightarrow q \land\ q \rightarrow p$$

$p \rightarrow q$: *if p then q*   : *p only if q*
$q \rightarrow p$: *if q then p*   : *p if q*

$p \leftrightarrow q$: *p if and only if q*
or,
$p \leftrightarrow q$: *p iff q*
       : *q iff p*

$p \leftrightarrow q \equiv p \rightarrow q \wedge q \rightarrow p$

$p \rightarrow q$: *p is sufficient condition for q*

$q \rightarrow p$: *p is necessary condition for q*

$p \leftrightarrow q$: *p is necessary and sufficient*
  condition for q

Or, since $p \leftrightarrow q \equiv q \leftrightarrow p$,

q is necessary and sufficient condition for p

Ex:

This program is correct if, and only if, it produces the correct answer for all possible sets of input data.

This is equivalent to:

If this program is correct, then it produces the correct answer for all possible sets of input data

and

if it produces the correct answer for all possible sets of input data then this program is correct.

This is equivalent to:

This program is correct is necessary and sufficient condition for it to produce correct answer for all possible sets of input data.

This is also equivalent to:

This program produces the correct answer for all possible sets of input data is necessary and sufficient condition for the program to be correct