Section 6.4
Closures of Relations

Definition: The *closure* of a relation R with respect to property P is the relation obtained by adding the minimum number of ordered pairs to R to obtain property P.

In terms of the digraph representation of R

- To find the reflexive closure - add loops.

- To find the symmetric closure - add arcs in the opposite direction.

- To find the transitive closure - if there is a path from a to b, add an arc from a to b.

Note: Reflexive and symmetric closures are easy. Transitive closures can be very complicated.

Definition: Let A be a set and let $\Delta = \{<x, x> \mid x \text{ in } A\}$. Δ is called the *diagonal relation* on A (sometimes called the *equality* relation E).

Note that D is the smallest (has the fewest number of ordered pairs) relation which is reflexive on A.

Reflexive Closure

Theorem: Let R be a relation on A. The *reflexive closure* of R, denoted r(R), is $R \quad \Delta$.

- Add loops to all vertices on the digraph representation of R.

- Put 1’s on the diagonal of the connection matrix of R.

Symmetric Closure

Definition: Let R be a relation on A. Then R^{-1} or the *inverse* of R is the relation $R^{-1} = \{< y,x > | < x,y > \quad R\}$

Note: to get R^{-1}

- reverse all the arcs in the digraph representation of R

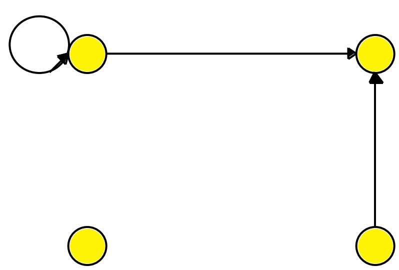
- take the transpose M^T of the connection matrix M of R.

Note: This relation is sometimes denoted as R^T or R^c and called the *converse* of R

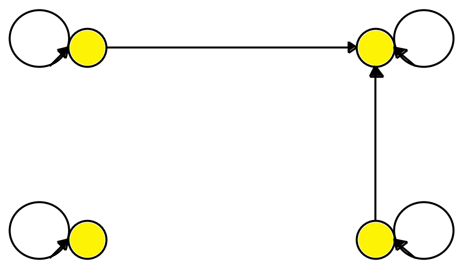
The composition of the relation with its inverse does not necessarily produce the diagonal relation (recall that the composition of a bijective <u>function</u> with its inverse is the identity).

Theorem: Let R be a relation on A. The *symmetric closure* of R, denoted s(R), is the relation $R \quad R^{-1}$.

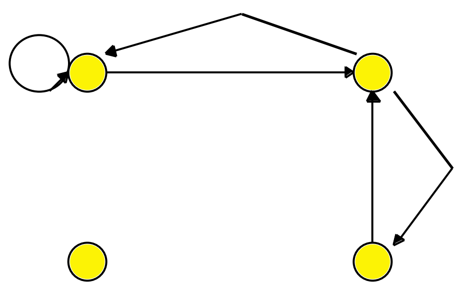
Examples:



R



r(R)



s(R)

Examples:

- If $A = Z$, then r(≠) = $Z \times Z$
- If $A = Z^+$, then s(<) = ≠.

  What is the (infinite) connection matrix of s(<)?

- If $A = Z$, then s(≤) = ?

Theorem: Let R_1 and R_2 be relations from A to B. Then

- $(R^{-1})^{-1} = R$
- $(R_1 \quad R_2)^{-1} = R_1^{-1} \quad R_2^{-1}$
- $(R_1 \cap R_2)^{-1} = R_1^{-1} \cap R_2^{-1}$
- $(A \times B)^{-1} = B \times A$
- $\quad ^{-1} =$
- $\overline{R}^{-1} = \overline{R^{-1}}$
- $(R_1 - R_2)^{-1} = R_1^{-1} - R_2^{-1}$
- If $A = B$, then $(R_1R_2)^{-1} = R_2^{-1} R_1^{-1}$
- If $R_1 \quad R_2$ then $R_1^{-1} \quad R_2^{-1}$

Theorem: R is symmetric iff $R = R^{-1}$

Paths

Definition: A *path* of *length n* in a digraph G is a sequence of edges $<x_0, x_1><x_1, x_2> \ldots <x_{n-1}, x_n>$.

The terminal vertex of the previous arc matches with the initial vertex of the following arc.

If $x_0 = x_n$ the path is called a *cycle* or *circuit*. Similarly for relations.

Theorem: Let R be a relation on A. There is a path of length *n* from *a* to *b* iff $<a, b> \quad R^n$.

Proof: (by induction)

• *Basis*: An arc from *a* to *b* is a path of length 1 which is in $R^1 = R$. Hence the assertion is true for $n = 1$.

• *Induction Hypothesis*: Assume the assertion is true for *n*.

Show it must be true for $n+1$.

There is a path of length n+1 from a to b iff there is an x in A such that there is a path of length 1 from *a* to *x* and a path of length *n* from *x* to *b*.

From the Induction Hypothesis,

$$<a, x> \quad R$$

and since <x , b> is a path of length n,

$$<x, b> \quad R^n.$$

If

$$<a, x> \quad R$$

and

$$<x, b> \quad R^n,$$

then

$$<a, b> \quad R^n \circ R = R^{n+1}$$

by the inductive definition of the powers of R.

Q. E. D.

Useful Results for Transitive Closure

Theorem:

If $A \quad B$ and $C \quad B$, then $A \quad C \quad B$.

Theorem:

If $R \quad S$ and $T \quad U$ then $R \circ T \quad S \circ U$.

Corollary:

If $R \quad S$ then $R^n \quad S^n$

Theorem:

If R is transitive then so is R^n

Trick proof: Show $(R^n)^2 = (R^2)^n \quad R^n$

Theorem: If $R^k = R^j$ for some $j > k$, then $R^{j+m} = R^n$ for some $n \leq j$.

We don't get any new relations beyond R^j.

As soon as you get a power of R that is the same as one you had before, STOP.

Transitive Closure

Recall that the transitive closure of a relation R, $t(R)$, is the smallest transitive relation containing R.

Also recall

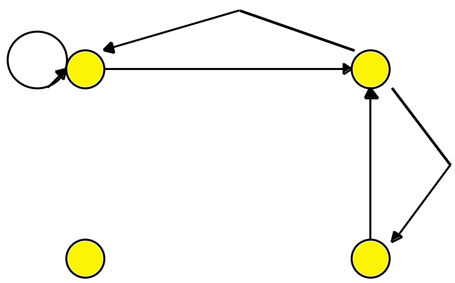
R is transitive iff R^n is contained in R for all n

Hence, if there is a path from x to y then there must be an arc from x to y, or $<x, y>$ is in R.

Example:

- If $A = Z$ and $R = \{< i, i+1>\}$ then $t(R) = <$

- Suppose R: is the following:



What is t(R)?

Definition: The *connectivity* relation or the *star closure* of the relation *R*, denoted *R*,* is the set of ordered pairs <*a*, *b*> such that there is a path (in *R*) from *a* to *b*:

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

Examples:

• Let $A = Z$ and $R = \{<i, i+1>\}$. $R^* = <$.

• Let *A* = the set of people, $R = \{<x, y> \mid$ person x is a parent of person $y\}$. $R^* = ?$

Theorem: $t(R) = R^*$.

Proof:

Note: this is not the same proof as in the text.

We must show that R^*

1) is a transitive relation

2) contains R

3) is the smallest transitive relation which contains R

Proof:

Part 2):

Easy from the definition of R^*.

Part 1):

Suppose $<x, y>$ and $<y, z>$ are in R^*.

Show $<x, z>$ is in R^*.

By definition of R^*, $<x, y>$ is in R^m for some m and $<y, z>$ is in R^n for some n.

Then $<x, z>$ is in $R^n R^m = R^{m+n}$ which is contained in R^*. Hence, R^* must be transitive.

Part 3):

Now suppose S is any transitive relation that contains R.

We must show S contains R^* to show R^* is the smallest such relation.

R S so R^2 S^2 S since S is transitive

Therefore R^n S^n S for all n. (why?)

Hence S must contain R^* since it must also contain the union of all the powers of R.

Q. E. D.

In fact, we need only consider paths of length n or less.

Theorem: If $|A| = n$, then any path of length $> n$ must contain a cycle.

Proof:

If we write down a list of more than n vertices representing a path in R, some vertex must appear at least twice in the list (by the Pigeon Hole Principle).

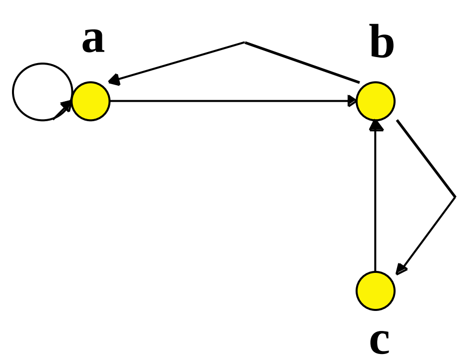
Thus R^k for $k > n$ doesn't contain any arcs that don't already appear in the first n powers of R.

Corollary: If | A | = n, then t(R) = R* = R R^2 . . . R^n

Corollary: We can find the connection matrix of t(R) by computing the join of the first n powers of the connection matrix of R.

Powerful Algorithm!

Example:



Do the following in class:

R2:

R3:

R4:

R5:

•

•

•

t(R) = R*:

So that you don't get bored, here are some problems to discuss on your next blind date:

1) Do the closure operations commute?

- Does st(R) = ts(R)?
- Does rt(R) = tr(R)?
- Does rs(R) = sr(R)?

2) Do the closure operations distribute

- Over the set operations?
- Over inverse?
- Over complement?
- Over set inclusion?

Examples:

- Does t(R1 - R2) = t(R1) - t(R2)?
- Does $r(R^{-1}) = [r(R)]^{-1}$?
