Section 6.3
Representing Relations

Connection Matrices

Let R be a relation from

\[ A = \{a_1, a_2, \ldots, a_m\} \]

to

\[ B = \{b_1, b_2, \ldots, b_n\}. \]

**Definition:** An \( m \times n \) connection matrix \( M \) for R is defined by

\[ M_{ij} = 1 \text{ if } <a_i, b_j> \text{ is in } R, \]
\[ = 0 \text{ otherwise.} \]

Example:

We assume the rows are labeled with the elements of A and the columns are labeled with the elements of B.

Let

\[ A = \{a, b, c\} \]

\[ B = \{e, f, g, h\} \]

\[ R = \{<a,e>, <c, g>\} \]
Then the connection matrix $M$ for $R$ is

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
$$

Note: the order of the elements of $A$ and $B$ matters.

**Theorem:** Let $R$ be a binary relation on a set $A$ and let $M$ be its connection matrix. Then

- $R$ is reflexive iff $M_{ii} = 1$ for all $i$.
- $R$ is symmetric iff $M$ is a symmetric matrix: $M = M^T$
- $R$ is antisymetric if $M_{ij} = 0$ or $M_{ji} = 0$ for all $i \neq j$.

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**Combining Connection Matrices**

**Definition:** the *join* of two matrices $M_1$, $M_2$, denoted $M_1 \lor M_2$, is the component wise boolean ‘or’ of the two matrices.

**Fact:** If $M_1$ is the connection matrix for $R_1$ and $M_2$ is the connection matrix for $R_2$ then the join of $M_1$ and $M_2$, $M_1 \lor M_2$, is the connection matrix for $R_1 \cup R_2$. 
**Definition:** the meet of two matrices $M_1, M_2$, denoted $M_1 \wedge M_2$ is the componentwise boolean ‘and’ of the two matrices.

**Fact:** If $M_1$ is the connection matrix for $R_1$ and $M_2$ is the connection matrix for $R_2$ then the meet of $M_1$ and $M_2$, $M_1 \wedge M_2$ is the connection matrix for $R_1 \cap R_2$.

Obvious questions:

Given the connection matrix for two relations, how does one find the connection matrix for

- The complement?
- The relative complement?
- The symmetric difference?

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**The Composition**

**Definition:** Let

$M_1$ be the connection matrix for $R_1$ and

$M_2$ be the connection matrix for $R_2$. 

The boolean product of two connection matrices \( M_1 \) and \( M_2 \), denoted \( M_1 \otimes M_2 \), is the connection matrix for the composition of \( R_2 \) with \( R_1 \), \( R_2 \circ R_1 \).

\[
(M_1 \otimes M_2)_{ij} = \bigvee_{k=1}^{n} [(M_1)_{ik} \land (M_2)_{kj}]
\]

Why?

In order for there to be an arc \(<x, z>\) in the composition then there must be an arc \(<x, y>\) in \( R_1 \) and an arc \(<y, z>\) in \( R_2 \) for some \( y \)!

The Boolean product checks all possible \( y \)'s. If at least one such path exists, that is sufficient.

Note: the matrices \( M_1 \) and \( M_2 \) must be conformable: the number of columns of \( M_1 \) must equal the number of rows of \( M_2 \).

If \( M_1 \) is \( mxn \) and \( M_2 \) is \( nxp \) then \( M_1 \otimes M_2 \) is \( mxp \).
Example:

\[
M_1 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]

\[
M_2 = \begin{bmatrix}
0 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 1
\end{bmatrix}
\]

\[
M_1 \otimes M_2 = \begin{bmatrix}
0 & 1 \\
0 & 1 \\
0 & 1
\end{bmatrix}
\]

\[
(M_1 \otimes M_2)_{12} = [(M_1)_{11} \land (M_2)_{12}] \lor [(M_1)_{12} \land (M_2)_{22}] \lor [(M_1)_{13} \land (M_2)_{32}] \lor [(M_1)_{14} \land (M_2)_{42}]
\]
\[ [0 \land 0] \lor [1 \land 1] \lor [0 \land 0] \lor [0 \land 1] = 1 \]

Note:

- there is an arc in \( R_1 \) from node 1 in A to node 2 in B
- there is an arc in \( R_2 \) from node 2 in B to node 2 in C.
- Hence there is an arc in \( R_2 \circ R_1 \) from node 1 in A to node 2 in C.

A useful result:

\[ M_{R^n} = M_R^n \]

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**Digraphs**

(see section 6.1)

Given the digraphs for \( R_1 \) and \( R_2 \), find the digraphs for

- \( R_2 \cup R_1 \)
- \( R_2 \cap R_1 \)
- \( R_2 - R_1 \)
• \( R_2 \oplus R_1 \)

• \( \overline{R}_1 \)