Section 1.2
Propositional Equivalences

A *tautology* is a proposition which is always true.

Classic Example: \( P \lor \neg P \)

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A *contradiction* is a proposition which is always false.

Classic Example: \( P \land \neg P \)

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A *contingency* is a proposition which neither a tautology nor a contradiction.

Example: \((P \lor Q) \rightarrow \neg R\)

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Two propositions \( P \) and \( Q \) are *logically equivalent* if \( P \leftrightarrow Q \) is a tautology. We write

\[ P \leftrightarrow Q \]

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Example: \((P \rightarrow Q) \land (Q \rightarrow P) \Leftrightarrow (P \leftrightarrow Q)\)
Proof:

The left side and the right side must have the same truth values independent of the truth value of the component propositions.

To show a proposition is not a tautology: use an abbreviated truth table

- try to find a counter example or to disprove the assertion.

- search for a case where the proposition is false

Case 1: Try left side false, right side true

Left side false: only one of $P \rightarrow Q$ or $Q \rightarrow P$ need be false.

1a. Assume $P \rightarrow Q = F$. Then $P = T$, $Q = F$. But then right side $P \leftrightarrow Q = F$. Oops, wrong guess.

1b. Try $Q \rightarrow P = F$. Then $Q = T$, $P = F$. Then $P \leftrightarrow Q = F$. Another wrong guess.

Case 2. Try left side true, right side false
If right side is false, P and Q cannot have the same truth value.

2a. Assume P = T, Q = F.
Then \( P \rightarrow Q = F \) and the conjunction must be false so the left side cannot be true in this case. Another wrong guess.

2b. Assume Q = T, P = F.
Again the left side cannot be true.

We have exhausted all possibilities and not found a counterexample. The two propositions must be logically equivalent.

Note: Because of this equivalence, if and only if or iff is also stated as is a necessary and sufficient condition for.

Some famous logical equivalences:

**Logical Equivalences**

\[
\begin{align*}
P \land T & \iff P \\
P \lor F & \iff P \\
P \lor T & \iff T \\
P \land F & \iff F \\
P \lor P & \iff P \\
P \land P & \iff P \\
\neg(\neg P) & \iff P \\
P \lor Q & \iff Q \lor P \\
P \land Q & \iff Q \land P \\
(P \lor Q) \land R & \iff P \lor (Q \land R) \\
(P \land Q) \lor R & \iff P \land (Q \lor R)
\end{align*}
\]
\[ P \land (Q \lor R) \iff \quad \text{Distributivity} \]
\[ (P \land Q) \lor (P \land R) \]
\[ P \lor (Q \land R) \iff \]
\[ (P \lor Q) \land (P \lor R) \]
\[ \neg (P \land Q) \iff \neg P \lor \neg Q \quad \text{DeMorgan’s laws} \]
\[ \neg (P \lor Q) \iff \neg P \land \neg Q \]
\[ P \to Q \iff \neg P \lor Q \quad \text{Implication} \]
\[ P \lor \neg P \iff T \quad \text{Tautology} \]
\[ P \land \neg P \iff F \quad \text{Contradiction} \]
\[ P \lor F \iff P \quad \text{Tautology} \]
\[ (P \to Q) \land (Q \to P) \iff \quad \text{Equivalence} \]
\[ (P \iff Q) \quad \text{Implication} \]
\[ (P \to Q) \land (P \to \neg Q) \iff \quad \text{Absurdity} \]
\[ \neg P \]
\[ (P \to Q) \iff (\neg Q \to \neg P) \quad \text{Contrapositive} \]
\[ P \lor (P \land Q) \iff P \quad \text{Absorption} \]
\[ P \land (P \lor Q) \iff P \quad \text{Absorption} \]
\[ (P \land Q) \to R \iff \quad \text{Exportation} \]
\[ P \to (Q \to R) \]

Note: equivalent expressions can always be substituted for each other in a more complex expression - useful for simplification.
Normal or Canonical Forms

Unique representations of a proposition

Examples:

Construct a simple proposition of two variables which is true only when

- P is true and Q is false:
  \[ P \land \neg Q \]

- P is true and Q is true:
  \[ P \land Q \]

- P is true and Q is false or P is true and Q is true:
  \[ (P \land \neg Q) \lor (P \land Q) \]

A disjunction of conjunctions where

- every variable or its negation is represented once in each conjunction (a minterm)

- each minterms appears only once

Disjunctive Normal Form (DNF)

Important in switching theory, simplification in the design of circuits.
Method: To find the minterms of the DNF.

- Use the rows of the truth table where the proposition is 1 or True
- If a zero appears under a variable, use the negation of the propositional variable in the minterm
- If a one appears, use the propositional variable.

Example:

Find the DNF of \((P \lor Q) \rightarrow \neg R\)

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>((P \lor Q) \rightarrow \neg R)</th>
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<tbody>
<tr>
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There are 5 cases where the proposition is true, hence 5 minterms. Rows 1, 2, 3, 5 and 7 produce the following disjunction of minterms:
\[(P \lor Q) \rightarrow \neg R\]

\[\iff (\neg P \land \neg Q \land \neg R) \lor (\neg P \land \neg Q \land R) \lor (\neg P \land Q \land \neg R) \lor (P \land \neg Q \land \neg R) \lor (P \land Q \land \neg R)\]

Note that you get a *Conjunctive Normal Form* (CNF) if you negate a DNF and use DeMorgan’s Laws.