# CS 512: Linear Programs and Duality

16:198:512
Instructor: Wes Cowan

A 'standard form' linear program usually looks like:

$$\begin{aligned} &\max_x \ c^{\mathrm{T}} x \\ &\text{s.t. } x \geq 0 \\ &Ax \leq b, \end{aligned} \tag{1}$$

where $x$ and $c$ are $n$-dimensional vectors, $b$ is an $m$-dimensional vector, and $A$ is an $m$ x $n$ dimensional matrix. The corresponding standard 'dual' is given by:

$$\begin{aligned} &\min_x \ y^{\mathrm{T}} b \\ &\text{s.t. } y \geq 0 \\ &y^{\mathrm{T}} A \geq c. \end{aligned} \tag{2}$$

As a specific example to explore the concepts of duality, we consider the following specific example:

$$\begin{aligned} &\max_{x_1,x_2,x_3} \ 1x_1 + 2x_2 + 3x_3 \\ &\text{s.t. } x_1, x_2, x_3 \geq 0 \\ &x_1 + x_2 \leq 30 \\ &x_2 + x_3 \leq 20 \end{aligned} \tag{3}$$

with the corresponding dual

$$\begin{aligned} &\min_{y_1,y_2} \ 30y_1 + 20y_2 \\ &\text{s.t. } y_1, y_2 \geq 0 \\ &y_1 \geq 1 \\ &y_1 + y_2 \geq 2 \\ &y_2 \geq 3. \end{aligned} \tag{4}$$

Where does this come from? Why is this here? Consider attempting to re-combine the constraints of the primal to try to get a bound on the objective function. For instance, for any feasible $x = (x_1, x_2, x_3)$, we must have that $(x_1+x_2)+3(x_2+x_3) \leq 30+3*20 = 90$, or $x_1+4x_2+3x_3 \leq 90$. Since we have that $x_1+2x_2+3x_3 \leq x_1+4x_2+3x_3$, this tells us that for any feasible $x$, we must have that the value of the objective function is at most 90. It is reasonable to ask - can we do better? Can we achieve tighter upper bounds?

Consider multiplying the first constraint by some factor $y_1$, to get $y_1(x_1 + x_2) \leq 30y_1$. Note, for this to hold, we need that $y_1 \geq 0$. Similarly, for the second constraint, we introduce a multiplier of $y_2 \geq 0$ to get $y_2(x_2 + x_3) \leq 20y_2$. Combining them,

$$y_1(x_1 + x_2) + y_2(x_2 + x_3) \leq 30y_1 + 20y_2 \tag{5}$$

or

$$(y_1)x_1 + (y_1 + y_2)x_2 + y_2x_3 \leq 30y_1 + 20y_2. \tag{6}$$

If it is additionally true that $y_1 \geq 1$, that $y_1 + y_2 \geq 2$ and $y_2 \geq 3$, we have immediately that

$$x_1 + 2x_2 + 3x_3 \leq (y_1)x_1 + (y_1 + y_2)x_2 + y_2x_3 \leq 30y_1 + 20y_2. \tag{7}$$

To summarize: for an *primal feasible* $x$, the value of the objective function at that $x$ can be bound from above by $30y_1 + 20y_2$ for any *dual feasible* $y$. The constraints of the dual come from ensuring that the objective function of the primal is bound from above. We can then ask, what is the smallest possible value of $30y_1 + 20y_2$ for any such dual feasible $y$? This is the 'tightest upper bound' on the maximal primal solution.

The duality theorem in linear programming says that if the objective function of the primal problem is bounded, the in fact this tightest upper bound exactly matches the true optimal value of the primal - there is no gap between the actual value and this dual upper bound.

In this particular case, the dual is easy to solve, plotting the feasible solution space and looking at the vertices; in particular, taking $y_1^* = 1$ and $y_2^* = 3$ minimizes the values of all variables, and satisfies all three constraints. This gives an optimum of $30 + 3 * 20 = 90$, and tells us that our initial bound discussed above was as good as it gets - by the duality theorem (why is the objective of the primal bounded?), we may conclude that the optimal value of the primal is 90.

At this point, we have that the solution to the dual is $y_1^* = 1$ and $y_2^* = 3$, but how can we reconstruct the solution to the primal?

Suppose that $(x_1^*, x_2^*, x_3^*)$ is the solution to the primal. Looking back at the connection between the two, $y^*$ as multipliers on the bounds, we have that

$$x_1^* + 2x_2^* + 3x_3^* = 90 = 30y_1^* + 20y_2^*. \tag{8}$$

In one sense, this equation reduces the number of variables we need to solve for ( as we can solve for $x_3^*$ in terms of $x_1^*$ and $x_2^*$). But we can actually go one step further: recall that for any primal feasible $x$ and dual feasible $y$ we have

$$x_1 + 2x_2 + 3x_3 \leq y_1(x_1 + x_2) + y_2(x_2 + x_3) \leq 30y_1 + 20y_2, \tag{9}$$

and hence now

$$90 = x_1^* + 2x_2^* + 3x_3^* \leq y_1^*(x_1^* + x_2^*) + y_2^*(x_2^* + x_3^*) \leq 30y_1^* + 20y_2^* = 90, \tag{10}$$

This can only be realized if $x_1^* + x_2^*$ is *exactly* 30, and $x_2^* + x_3^*$ is *exactly* 20, that is the two constraints are realized with equality. This gives us the following system of equations:

$$\begin{aligned} x_1^* + x_2^* &= 30 \\ x_2^* + x_3^* &= 20. \end{aligned} \tag{11}$$

Consider solving this system in terms of $x_2^*$: $x_1^* = 30 - x_2^*$ and $x_3^* = 20 - x_2^*$. Substituting in, we have

$$x_1^* + 2x_2^* + 3x_3^* = 30 - x_2^* + 2x_2^* + 3(20 - x_2^*) = 90 - x_2^* + 2x_2^* = 90 - x_2^*. \tag{12}$$

It's clear from this that the solution of the primal is given by $x = (30, 0, 20)$.

How to generalize this? Recall the correspondence between variables and constraints in the primal and the dual. For any $y_i$ in the dual and its corresponding constraint in the primal, if $y_i^* > 0$, then that constraint must be realized *with equality* in the solution to the primal. So from the solution to the dual, you can recover a system of equations from the constraints of the dual to help solve for $x^*$.

However, this is not everything: noting again the correspondence between variables and constraints, observe that the second constraint of the dual was not realized with equality - there was *slack* in the solution. The first and the third constraints had no slack, however. By the same principle, any variable in the primal that corresponds to a *slack constraint* in the dual can be taken to be zero - hence $x_2^* = 0$.

This correspondence is known as complementary slackness - for the optimal solutions to the primal and dual (when they exist) positive variables correspond to constraints realized with equality, and constraints realized with strict inequality correspond to variables equal to zero. This principle can be used to convert dual solutions into primal solutions and vice versa, by generating a system of equations (and zero-variables) to solve and yield the solution.