

## CS 520: CoinBot

16:198:520

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This is meant to be a companion piece to the writeup of Probabilistic Knowledge and Queries.

## 1 Preliminaries

- 0) Argue that if a coin has a probability of heads  $p$ , then out of  $n$  independent flips, if  $X$  is the number of heads, then

$$\mathbf{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}. \quad (1)$$

*One approach: What events can you consider marginalizing on? Also consider: what are you taking as the 'givens' in this computation?*

## 2 The Coin Bot

You are given a mysterious coin. It is either Coin A, which has a probability of 0.4 of giving heads and 0.6 of giving tails, or it is Coin B, which has a probability of 0.7 of giving heads and 0.3 of giving tails. Initially, you have no reason to think that either coin is more likely than the other (i.e., uniform priors). However, by flipping the coin multiple times, you plan on using this data to reason about which coin it is more likely to be,  $A$  or  $B$ .

Let  $F_1, F_2, F_3, \dots, F_n$  be the sequence of flips collected through time  $n$  (i.e.,  $F_t = 0$  means the  $t$ th flip was tails,  $F_t = 1$  means the  $t$ th flip was heads). We may quantify our belief at any given point in time with the two functions

$$\begin{aligned} p_A(n) &= \mathbf{P}(\text{Coin} = A | F_1, F_2, \dots, F_n) \\ p_B(n) &= \mathbf{P}(\text{Coin} = B | F_1, F_2, \dots, F_n). \end{aligned} \quad (2)$$

- 1) Prove that, for any  $n$ ,  $p_A(n) + p_B(n) = 1$ .
- 2) Prove that for any  $n$ ,  $p_A(n)$  and  $p_B(n)$  depend only on *the number of heads and tails recorded* rather than the specific order of flips collected. Give an explicit formula for both in terms of the respective probabilities, and the total number of heads recorded in  $n$  flips,  $\text{Heads}_n$ .

Given a set of data  $F_1, \dots, F_n$ , we can construct the following decision rule for guessing what the coin is: let  $\text{Guess}_n = A$  if  $p_A(n) \geq p_B(n)$ , or take  $\text{Guess}_n = B$  if  $p_B(n) > p_A(n)$ .

- 3) Simplify the test  $p_B(n) > p_A(n)$  as much as possible to a simple test on the value of  $\text{Heads}_n$ .
- 4) What is  $\mathbf{P}(\text{Guess}_n = B | \text{Coin} = A)$ ? That is, if the coin were *actually*  $A$ , what is the probability that after  $n$  flips, we would guess that the coin is  $B$ ? What is  $\mathbf{P}(\text{Guess}_n = A | \text{Coin} = B)$ ? *Hint: What can / should you marginalize on?* Give answers for  $n = 5, 10, 100$ .
- 5) Combining the results of the previous question, what is

$$\mathbf{P}(\text{Guess}_n \neq \text{Coin})? \quad (3)$$

*What should you marginalize on? What are your priors?* Give answers for  $n = 5, 10, 100$ .

- 6) What is the smallest  $n$  such that  $\mathbf{P}(\text{Guess}_n \neq \text{Coin}) \leq 0.1$ ? How does this inform how many times you ought to experiment by flipping?
- 7) The answer to the previous question can inform how many experiments you ought to do. But *once you have performed those experiments*, it is reasonable to ask - what is the probability that you guessed wrong? i.e., what is

$$\mathbf{P}(\text{Guess}_n \neq \text{Coin} | F_1, F_2, \dots, F_n) \text{ or } \mathbf{P}(\text{Guess}_n \neq \text{Coin} | \text{Heads}_n)? \quad (4)$$

*What should you marginalize on? Given the data, is  $\text{Guess}_n$  random?*

- 8) What value of  $\text{Heads}_n$  maximizes  $\mathbf{P}(\text{Guess}_n \neq \text{Coin} | \text{Heads}_n)$  as in the previous question? Initially, what is the probability of this value of  $\text{Heads}_n$  occurring? (*Marginalize!*)
- 9) Freeform question: Suppose that you know that the coin is Coin  $A$ , but after some (unknown) number of flips, someone is going to swap it for Coin  $B$  without telling you. Design a test CoinBot can use to try to determine when / if the coin has been swapped. What can you say about the probability of CoinBot declaring the coin has been swapped before it was actually swapped (false positives)? What can you say about how many flips it takes for CoinBot to realize the coin has been swapped?