Propositional Logic 1 – brief REVIEW  
(section 1.1)

- Propositions
- Connectives
  1. Negation \( \sim \)
  2. Conjunction \( \land \)
  3. Disjunction \( \lor \)
  4. Implication
  5. Biconditional

Precedence: Book says the order is 1 to 5 above. But the only thing there is agreement about in books is that \( \sim \) has higher precedence than \( \land \), which has higher precedence than \( \lor \). For everything else, use parentheses.

(more on implication)

- One way to view the logical conditional is to think of an obligation or contract.
  1. “If I am elected, then I will lower taxes.”
  2. “If you get 100% on the final, then you will get an A.”
- If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge. Something similar holds for the professor. This corresponds to the case where \( p \) is true and \( q \) is false.
- For an attempt at explanation by a philosopher, see http://www.earlham.edu/~peters/courses/log/mat-imp.htm

(Understanding Implication)

In \( p \rightarrow q \) there does not need to be any connection between the antecedent or the consequent. The “meaning” of \( p \rightarrow q \) depends only on the truth values of \( p \) and \( q \).

These implications are perfectly fine, but would not be used in ordinary English.

1. “If the moon is made of green cheese, then I have more money than Bill Gates.”
2. “If the moon is made of green cheese then I’m on welfare.”
3. “If \( 1 + 1 = 3 \), then your grandma wears combat boots.”

Different Ways of Expressing \( p \rightarrow q \) in English (read in book)

- if \( p \), then \( q \)  \( p \) implies \( q \)
- if \( p, q \)  \( p \) only if \( q \)
- \( q \) unless \( \neg p \)  \( q \) when \( p \)
- \( q \) if \( p \)  \( q \) when \( p \)
- \( q \) whenever \( p \)  \( p \) is sufficient for \( q \)
- \( q \) follows from \( p \)  \( q \) is necessary for \( p \)
- a necessary condition for \( p \) is \( q \)
- a sufficient condition for \( q \) is \( p \)
Truth tables – the first time

*TRUTH TABLES: show how to evaluate truth value of formulas, given truth value of their arguments (“truth functional composition”) \( \land \lor \)

\[
\begin{array}{cccccc}
p & q & p \lor q & p \land q & \neg p & \neg p \lor q & \text{true} \\
\hline
 t & t & t & t & f & t & t \\
t & f & f & t & f & f & t \\
f & t & f & t & t & t & t \\
f & f & f & f & t & t & t \\
\end{array}
\]

• Lines in truth tables correspond to ‘possible states of the world’

• Note that \( p \rightarrow q \) and \( \neg p \lor q \) have same truth table columns; called *logically equivalent*; notation: \( p \rightarrow q = \neg p \lor q \)

• Note that the logical constant symbol \texttt{true} also has truth table

(In case you find this useful: addition table as an analogy to truth table)

Say variable \( x \) can be \( \{0,1,2,3,\ldots\} \)? What are values for \( x+2 \)? For the constant 7?

<table>
<thead>
<tr>
<th>( x )</th>
<th>( x+2 )</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
<td>7</td>
</tr>
</tbody>
</table>

Propositional Logic - lecture 2

Lecture slides based on those provided by textbook web site. Almost all material here is in Sections 2.1 and 2.2

Terminology

A formula is said to be a

• **tautology** if its truth table column is all \texttt{true}

• **satisfiable/consistent** if at least one row of its truth table column is \texttt{true}

• **contradiction/inconsistent/unsatisfiable** if its truth table column is all \texttt{false}

• A set of formulas \( S \) is said to be **satisfiable/consistent** if the formula obtained by *conjoining* all the formulas in \( S \) is satisfiable

Proposition: A formula \( R \) is **unsatisfiable** if and only if \( \neg R \) is a **tautology**.
Translating English Sentences – review

- Steps to convert an English sentence to a statement in propositional logic
  1. Identify atomic propositions and represent using propositional variables.
  2. Determine appropriate logical connectives
- “If I go to Harry’s or to the country, I will not go shopping.”
  1. p: I go to Harry’s
  2. q: I go to the country.
  3. r: I will go shopping.
  “If p or q then not r.”
  \((p \lor q) \rightarrow \neg r\)

(home exercise)

- Problem: Translate the following sentence into propositional logic:
  “You can access the Internet from campus only if you are a computer science major or you are not a freshman.”
- One Solution: Let a, c, and f represent respectively “You can access the internet from campus,” “You are a computer science major,” and “You are a freshman.”
  \(a \rightarrow (c \lor \neg f)\)

Applications of PropCalc

The book has others. I’ll give you different ones.
Larger example of GORE

The following example shows part of a much larger example for the Italian Trentino Ministry of Culture. It has additional notation for edges labelled with $+,++,-$ and $--$. These represent influences that are not captured by standard logical implication; they are more like "if A is the case then there is a weak/strong likelihood that B is (not) the case."

This general notation is known as $i^*$, and Eric Yu & John Mylopoulos are its originators, though now there is a whole community working on it.
2. Map coloring problem

Suppose map has 5 countries. Want to know if we can color them with exactly 3 colors

Need propositions for:
- green, red, blue, colored, nextTo, sameColor

Axioms
- colored(c) ←→ green(c) ∨ red(c) ∨ blue(c)
- sameColor(x,y) ←→ green(x)\green(y) ∨ red(x)\red(y) ∨ blue(x)\blue(y)

/* Describe uncolored map */
nextTo(1,2). nextTo(1,3). nextTo(2,4)... nextTo(4,5).
/* Neighbors cannot be colored same */
nextTo(x,y) → ¬sameColor(x,y)
/* All nodes must be colored */
colored(1). colored(2). ... colored(5).

De Morgan’s Laws and how one proves them

\neg(p \land q) \equiv \neg p \lor \neg q \\
\neg(p \lor q) \equiv \neg p \land \neg q

We use a truth table to show that De Morgan’s Second Law holds.

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>\neg p</th>
<th>\neg q</th>
<th>(p\lor q)</th>
<th>\neg(p\lor q)</th>
<th>\neg(p\land q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>
Other Key Logical Equivalences

(Onces in black are often skipped, as "obvious")

- **Identity Laws:**  
  \[ p \land T \equiv p \quad p \lor F \equiv p \]

- **Domination Laws:**  
  \[ p \lor T \equiv T \quad p \land F \equiv F \]

- **Idempotent laws:**  
  \[ p \lor p \equiv p \quad p \land p \equiv p \]

- **Double Negation Law:**  
  \[ \neg(\neg p) \equiv p \]

- **Negation Laws:**  
  \[ p \lor \neg p \equiv T \quad p \land \neg p \equiv F \]

More Key Logical Equivalences

- **Commutative Laws:**  
  \[ p \land q \equiv q \land p \quad p \lor q \equiv q \lor p \]

- **Associative Laws:**  
  \[ (p \land q) \land r \equiv p \land (q \land r) \quad (p \lor q) \lor r \equiv p \lor (q \lor r) \]

- **Distributive Laws:**  
  \[ (p \land (q \lor r)) \equiv (p \land q) \lor (p \land r) \quad (p \lor (q \land r)) \equiv (p \lor q) \land (p \lor r) \]

- **Absorption Laws:**  
  \[ p \land (p \lor q) \equiv p \quad p \lor (p \land q) \equiv p \]

- **de Morgan’s law 1**  
  \[ \neg(p \lor q) \equiv \neg p \land \neg q \]

- **de Morgan’s law 2**  
  \[ \neg(p \land q) \equiv \neg p \lor \neg q \]

- **Definition of ->**  
  \[ p \to q \equiv \neg p \lor q \]

- **Definition of <->**  
  \[ p \leftrightarrow q \equiv (p \to q) \land (q \to p) \]

Summary of fundamental equivalences

[Table 6 extended/Sec 1.3 from textbook]

<table>
<thead>
<tr>
<th>p \land T = p</th>
<th>Identity Laws</th>
<th>(p \lor q) \land r = p \lor (q \lor r)</th>
<th>Associative laws (often omitted in proofs)</th>
<th>(p \land q) \lor r = p \land (q \lor r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>p \lor F = p</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>p \lor T = T</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>p \land F = F</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>p \lor p = p</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>p \land p = p</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>\neg(\neg p) = p</td>
<td>Double negation law</td>
<td>p \lor (p \land q) = p</td>
<td>Absorption laws</td>
<td>p \lor (p \lor q) = p</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>\neg(\neg p) = p</td>
<td>Double negation law</td>
<td>p \lor (p \land q) = p</td>
<td>Absorption laws</td>
<td>p \lor (p \lor q) = p</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>p \lor q \equiv q \lor p</td>
<td>Commutative laws (often omitted)</td>
<td>p \lor q \equiv q \lor p</td>
<td>Negation laws</td>
<td>p \lor q \equiv q \lor p</td>
</tr>
<tr>
<td>p \land q = q \land p</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>p \land q = q \land p</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>p \lor q = \neg p \lor q</td>
<td>Definition of -&gt;</td>
<td>p \lor q = (p \to q) \lor (q \to p)</td>
<td>Definition of &lt;-&gt;</td>
<td>p \lor q = (p \to q) \lor (q \to p)</td>
</tr>
</tbody>
</table>

Proofs using equivalences

(Remember in algebra, substitution of equals for equals)

RULES OF REASONING FOR equivalences:

a)  If A=B and B=C then A=C

b)  If

\[ \cdot \]  
\[ C \text{ is a formula}, \]  
\[ P \text{ is some } sub\text{-formula of } C, \]  
\[ Q \text{ is another formula such that } P=Q \]  
\[ D \text{ is the formula obtained by replacing } P \text{ by } Q \text{ in } C \]  
\[ \text{THEN } C == D \]

C) In any equivalence, a symbol can stand for a formula

Proofs then look like

1. p1 == q1 (given or from previous lines, with justification)

\[ \ldots \]

n. p_n == q_n (by replacing p_j with q_j, usually in the previous line)
Proof using Logical Equivalence – 1

**Show** \((p \rightarrow r) \lor (q \rightarrow r) \equiv (p \land q) \rightarrow r\)

\[
\begin{align*}
(p \rightarrow r) \lor (q \rightarrow r) & \equiv (\neg p \lor r) \lor (\neg q \lor r) \quad \text{Defn of } \lor \\
\equiv (\neg p \lor r) \lor (\neg q \lor r) & \quad \text{Assoc. } \lor \\
\equiv (\neg p \lor r) \lor (\neg q \lor r) & \quad \text{Comm. } \lor \\
\equiv (\neg p \lor r) \lor (\neg q \lor r) & \quad \text{Assoc. } \lor \\
\equiv (\neg (p \land q) \lor r) & \quad \text{De Morgan} \\
\equiv (p \land q) \lor (r) & \quad \text{Idempotent} \\
\equiv (p \land q) \rightarrow r & \quad \text{Defn of } \rightarrow 
\end{align*}
\]

Equivalence Proof – 2

**Show** \(\neg(p \lor (\neg p \land q))\) is logically equivalent to \(\neg p \land \neg q\)

**Solution:**

\[
\begin{align*}
\neg(p \lor (\neg p \land q)) & \equiv \neg p \land \neg(\neg p \land q) \quad \text{by DeMorgan} \\
& \equiv \neg p \land (p \lor \neg q) \quad \text{by DeMorgan inside} \\
& \equiv \neg p \land (p \lor \neg q) \quad \text{by double negation} \\
& \equiv (\neg p \land p) \lor (\neg p \land \neg q) \quad \text{by distributive} \\
& \equiv F \lor (\neg p \land \neg q) \quad \text{by negation law} \\
& \equiv (\neg p \land \neg q) \lor F \quad \text{by commutative} \\
& \equiv \neg p \land \neg q \quad \text{by identity} 
\end{align*}
\]

(In practice, one would skip the second to last step, assuming that all laws are duplicated with commuted counterparts)

Equivalence Proof – 3

- **Example:** Show \((p \land q) \rightarrow (p \lor q)\) is a tautology.
- **Solution:**

\[
\begin{align*}
(p \land q) \rightarrow (p \lor q) & \equiv \neg(p \land q) \lor (p \lor q) \quad \text{by def'n of } \rightarrow \\
& \equiv (\neg p \lor \neg q) \lor (p \lor q) \quad \text{by De Morgan} \\
& \equiv (\neg p \lor \neg q) \lor (p \lor q) \quad \text{by assoc. and comute} \\
& \equiv T \lor T \quad \text{by negation law (twice)} \\
& \equiv T \quad \text{by idempotent law (or by domination law)} 
\end{align*}
\]

Example (Prof. Kim)

- At a trial, it is stated that:
  1. “Sue is guilty and Jill is innocent.”
  2. “If Bill is guilty, then so is Jill.”
  3. “I am innocent, but at least one of the others is guilty.”
- Let \(b = \) Bill is innocent, \(j = \) Jill is innocent, and \(s = \) Sue is innocent
- Statements then are:
  1. \(\neg s \land j\)
  2. \(\neg b \rightarrow \neg f\)
  3. \(f \land (\neg b \lor \neg s)\)
- Can all the statements be true? (Yes iff their conjunction is not a contradiction, ie. /=F)
A propositional formula is in disjunctive normal form if it consists of a disjunction of \((1, \ldots, n)\) conjuncts where each conjunct consists of a conjunction of \((1, \ldots, m)\) literals (literal = atomic formula or the negation of an atomic formula).
- \((p \land \neg q) \lor (\neg p \lor q)\) YES
  - because a single literal is a degenerate case of a conjunction
- \(p \land (\neg p \lor q)\) NO

Disjunctive Normal Form is important for the circuit design methods discussed in Chapter 12.

Theorem: every compound proposition \(S\) can be put in disjunctive normal form.

Solution:
- Construct the truth table for the proposition \(S\).
- For each row where \(S\) is true, conjoin an entry for every propositional variables \(p_j\) as follows: if the value for \(p_j\) is T, then just conjoin \(p_j\); if the value for \(p_j\) is F, then conjoin \(\neg p_j\).
- The DNF corresponding to \(S\) is the OR (disjunction) of all the above conjunctions
Conjunctive Normal Form

A compound proposition is in Conjunctive Normal Form (CNF) if it is a conjunction of disjunctions.

Every proposition can be put in an equivalent CNF.

Conjunctive Normal Form (CNF) can be obtained by eliminating implications, moving negation inwards and using the distributive and associative laws.

Important in resolution theorem proving and satisfiability testing, used in artificial intelligence (AI), system verification.

A compound proposition can be put in conjunctive normal form through repeated application of the logical equivalences covered earlier.

Example:

Put the following into CNF:

Solution:

1. Eliminate implication signs:

$$\neg(p \rightarrow q) \lor (r \rightarrow p)$$

3. Move negation inwards; eliminate double negation:

$$\neg(p \lor q) \land (\neg p \lor r)$$

5. Convert to CNF using associative/distributive laws

$$(p \lor \neg r \lor p) \land (q \lor \neg r \lor p)$$

Propositional Satisfiability

A proposition is satisfiable if there is an assignment of truth values to its variables that make it true. When no such assignments exist, the compound proposition is unsatisfiable.

Theorem: A proposition is unsatisfiable if and only if its negation is a tautology.