Rules of inference for Predicate Calculus

**Existential Generalization (E.G.)**

\[
P(f) \quad \frac{}{\exists x. P(x)}
\]

**Example:**
Michelle got an A in the class.
Therefore, someone got an A in the class.
f can be any constant!

**Existential Instantiation (E.I.)**

\[
\exists x. P(x) \quad \frac{}{P(e)}
\]

**Example:**
There is someone who got a B in the course. \( \exists x. \text{GotB}(x) \)
Let’s call her \( e \) – she’s the one who got a B \( \text{GotB}(e) \)
Note that we cannot be sure this \( x \) is the same as an \( \exists x \) in any other formula, so we have to use a new name.

**Example:**
Exists \( x \). Pretty\( (x) \). Exists \( x \). Rich\( (x) \) wth domain Person

Note that we cannot be sure who the \( x \)’s are, so we must use different names, not used before: Pretty\( (e1) \). Rich\( (e2) \).
Universal Instantiation (U.I.) part 1

\[ \forall x P(x) \quad \text{some previously introduced } c \]
\[ \therefore P(c) \]

Example:
Domain consists of all persons. If jamil is a constant name then from

“All persons can talk.” \( \forall x. \text{Can talk}(x) \)

one can conclude

“Therefore, jamil can talk.” \( \text{Can talk}(\text{Jamil}) \)

But in fact the e1 and e2 introduced on the previous slides are also persons. So \( \text{Can talk}(e1), \text{Can talk}(e2) \)

Universal Generalization (U.G.)

\[ P(d) \]
\[ \text{provided } d \text{ is arbitrary.} \]
\[ \therefore \forall x.P(x) \]

Used often implicitly in Mathematical Proofs:
“choose an arbitrary d” ..... The fact that d is arbitrary allows us to generalize to universal quantifier, in a way in which from Happy(jamil) you should not be able to conclude that everyone is happy!

Universal Instantiation (U.I.) part 2

\[ \forall x P(x) \quad \text{for arbitrary new constant } c \text{ or some previously introduced } c \]
\[ \therefore P(c) \]

Example: given \( \forall x. \text{Can talk}(x) \)

Sometime we want to say “let c be an arbitrary person”. Then \( \text{Can talk}(c) \) follows. It is important that we start with nothing else known about this arbitrary c because of the U.G. rule (next slide)
What is a valid first order logic proof?

Any propositional proof (but now symbols A,B can be any first order formula, not just a propositional formula) + obtaining one line from the preceding one using one of the 4 rules of inference for quantifiers.

Here come lots of examples:

Example 0
Example: Using the rules of inference, construct a valid argument to show that "John Smith has two legs" is a consequence of the premises:
i. “Every man has two legs.”
ii. “John Smith is a man.”
Solution: Let $M(x)$ denote “$x$ is a man” and $L(x)$ “$x$ has two legs” and let $j$ be a member of the domain standing for John Smith.
Valid Argument:

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
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<tbody>
<tr>
<td>1. $\forall x. (M(x) \rightarrow L(x))$</td>
<td>premise i</td>
</tr>
<tr>
<td>2. $M(j)$</td>
<td>premise ii</td>
</tr>
<tr>
<td>3. $M(j) \rightarrow L(j)$</td>
<td>U.I. from 1 (using existing j</td>
</tr>
<tr>
<td>4. $L(j)$</td>
<td>Modus ponens 2,3</td>
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Example 0.5 (a)
Try to prove:
$\forall x. P(x), \forall y. Q(y) \vdash \forall w. (P(w) \land Q(w))$

<table>
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<th>Step</th>
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<tbody>
<tr>
<td>1. $\forall x. P(x)$</td>
<td>given</td>
</tr>
<tr>
<td>2. $P(g)$</td>
<td>U.I. 1, arbitrary constant $g$</td>
</tr>
<tr>
<td>3. $\forall y. Q(y)$</td>
<td>given</td>
</tr>
<tr>
<td>4. $Q(g)$</td>
<td>U.I. 1, previously used constant $g$</td>
</tr>
<tr>
<td>5. $P(g) \land Q(g)$</td>
<td>Addition 2,4</td>
</tr>
<tr>
<td>6. $\forall w. (P(w) \land Q(w))$</td>
<td>U.G. 5 since $g$ is arbitrary</td>
</tr>
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Example 0.5 (b)
Try to prove:
$\exists x. P(x), \exists y. Q(y) \vdash \exists w. (P(w) \land Q(w))$

This would mean that whenever the stuff before $\vdash$ (lhs) is true then the stuff after $\vdash$ (rhs) is true; this does not seem to be the case: imagine $P(x) =$ “green $x$”, $Q(x) =$ “hungry $x$”. Just because there is someone green, and there is someone hungry, it doesn’t mean that it has to be the same person.

So construct the counter example:
$U = \{joe, ann\}$
$Facts = \{P(joe), Q(ann)\}$
These make both $\exists x. P(x)$ and $\exists y. Q(y)$ true, but not $\exists w. (P(w) \land Q(w))$
Verify that smaller universe or fewer facts do not have this property.
Example 1

Example 1: Use the rules of inference to construct a valid argument showing that the conclusion
"Someone who passed the first exam has not read the book."
follows from the premises
i. “A student in this class has not read the book.”
ii. “Everyone in this class passed the first exam.”

Solution: Let $C(x)$ denote “$x$ is in this class,” $R(x)$ denote “$x$ has read the book,” and $P(x)$ denote “$x$ passed the first exam.”

First we translate the premises and conclusion into symbolic form.

$\exists x. (C(x) \land \neg R(x))$, $\forall x. (C(x) \to P(x))$

|$\neg \exists x. (P(x) \land \neg R(x))$

Continued on next slide

Example 1 Proof using NatDedn rules

1 $\exists x. (C(x) \land \neg R(x))$ given premise
2 $C(e) \land \neg R(e)$ E.I. (special new e)
3 $C(e)$ $\land$-elim 2
4 $\forall x. (C(x) \to P(x))$ given premise
5 $C(e) \to P(e)$ U.I. 4 (e previous constant).

we could have replaced x by anything here but we chose e so 6. would work
6 $P(e)$
7 $\neg R(e)$
8 $P(e) \land \neg R(e)$ $\land$-intro 6,7
9 $\exists x. (P(x) \land \neg R(x))$ E.G. 8

In line 9 we could not have concluded $\forall x. (P(x) \land \neg B(x))$
by U.G. because e was not “arbitrary”, it was “special”!!!

Important observation

On the difference between
(i) $\exists x. P(x) \land \exists x. Q(x)$
(ii) $\exists x. (P(x) \land Q(x))$

• We can apply E.I. to (ii) to get $P(b) \land Q(b)$ for a special new constant $b$ because the formula is of the form $\exists x. \text{var} \ldots$

• We cannot apply E.I. to (i) because it’s top connective is $\land$, not $\exists$. So one can, at best, do something like
1 $\exists x. P(x) \land \exists x. Q(x)$
2 $\exists x. P(x)$ from 1 by Simplificn
3 $P(b)$ for special new $b$ (by E.I. from 2)
4 $\exists x. Q(x)$ from 1 by Simplificn
5 $Q(c)$ for special new $c$ ($c$ has to look different from $b$)
6 $P(b) \land Q(c)$ from 3,5 by Conjunction
Example 3

\[ \exists y. \ \forall x. \text{likes}(x,y) \vdash \ \forall w. \ \exists v. \text{likes}(w,v) \]

(note that intuitively this should be true: whoever is the value of \( y \), can be the value of all the \( v \)'s)

1. \( \exists y. \ \forall x. \text{likes}(x,y) \) given
2. \( \forall x. \text{likes}(x,a) \) \( \quad \) \text{E.I. 1, for special new } a
3. \text{likes}(b,a)  \quad \) \text{U.I. 2, for arbitrary new } b
4. \( \exists v. \text{likes}(b,v) \)  \( \quad \) \text{E.G. 3}
5. \( \forall w. \ \exists v. \text{likes}(w,v) \) \( \quad \) \text{U.G. 4 (because } b \text{ was arbitrary)}

Example 4

\[ \forall x. \ \exists y. \text{likes}(x,y) \vdash \ \exists w. \ \forall z. \text{likes}(w,z) \]

(note that intuitively this should NOT be true)

Construct counter example where \( U=\{a,b\} \) and there are different “likers”, so rhs is false, but lhs is true:

\[ \text{likes}(a,_) \]
\[ \text{likes}(b,_) \]

... so try Facts: \{ \text{likes}(b,a), \text{likes}(a,b) \} 

This makes \( \forall x. \ \exists y. \text{likes}(x,y) \) true but \( \exists w. \ \forall z. \text{likes}(w,z) \) false, as we wanted. Can we do it with fewer Facts? No, because otherwise lhs is false too, and then we do not have a counter example.