Efficient algorithms for tensor scaling, quantum marginals, and moment polytopes

Cole Franks (Rutgers)

joint work with

Peter Bürgisser, Ankit Garg, Rafael Oliveira,

Michael Walter, Avi Wigderson

Universities and institutes logos included.
Overview

Simple, efficient algorithm for *approximate* membership for broad class of polytopes

- known as **moment polytopes** in math and physics
- can have exponentially many facets and vertices
- capture many natural problems across computer science, mathematics, and physics
Outline

• Moment polytopes
• History/Applications
• Algorithms
• Analysis
• Open problems
Moment polytopes
Moment map

Generalization of momentum.

$$\mu : \text{manifold} \rightarrow \text{vector space}$$

Compatible with action of group $G$.
For us, $G = \text{GL}_{n_1}(\mathbb{C}) \times \ldots \times \text{GL}_{n_d}(\mathbb{C})$.

$$\mu : \begin{array}{c} \text{X} \\ \text{manifold} \end{array} \rightarrow \text{Herm}_{n_1} \times \cdots \times \text{Herm}_{n_d}.$$ 

For us X is projective variety (i.e. set of solutions to some homogeneous polynomials)
Expected value

\[ X \text{ vector space with basis vectors } v_\omega, \omega \in \Omega \subset \mathbb{Z}^n, \]
\[ G = n \times n \text{ diagonal matrices} \]
Action: \[ x \cdot v_\omega = x^{\omega} v_\omega. \]
If \( p \) is a probability distribution on \( \Omega \) and \( v = \sum_{\omega \in \Omega} \sqrt{p_\omega} v_\omega \),
\[ \mu : v \mapsto \mathbb{E}[\omega] \]
is a moment map for action of \( G \) on \( X \).
Takes values in Newton polytope of multivariate polynomial \( \sum_{\omega \in \Omega} x^{\omega} \).
Example: row and column sums

$X = \text{Mat}_{m \times n}(\mathbb{C})$, $G = D_1 \times D_2$ diagonal complex invertible matrices

$$(A, B) \cdot X = AXB^*$$

$\mu : X \to \mathbb{R}^{2n}$

$\mu(X) = \frac{1}{\sum |X_{ij}|^2}$ (square $\ell_2$ norms of rows, square $\ell_2$ norms of columns)

If $X_{ij} = \sqrt{Y_{ij}}$, the moment map is normalized row and column sums of $Y$. 
### Example: Matrix scaling

\[ G = D_1 \times D_2 \] diagonal complex invertible matrices \( X = \overline{G \cdot X} \)

**orbit closure** of the matrix \( X \)

All things obtainable from \( X \) by reweighting (scaling) row and columns

<table>
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<td>[ \Delta(X) := { \mu(Y) : Y \in X } ]</td>
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_is a convex polytope. In fact,_

| \[ \Delta(X) = \{ \mu(Y) : \text{supp } Y \subset \text{supp } X \} \] |

\( \Delta(X) \) example of a _moment polytope._
Example: Horn’s problem

Question:
What is the set \( \{ (\lambda(A + B), \lambda(A), \lambda(B)) : A, B \in \text{Herm}_n \} \)?

\[ \mathbf{X} = \text{Mat}_{n \times n} \times \text{Mat}_{n \times n}, \quad \mathbf{G} = \text{GL}_n \times \text{GL}_n \times \text{GL}_n \]

\[ \mu : \mathbf{X} \rightarrow \text{Herm}_n \times \text{Herm}_n \times \text{Herm}_n \]

\[ \mu(X, Y) = (X^*X + Y^*Y, XX^*, YY^*) \]

Surprisingly, \( \Delta(\mathbf{X}) = \{ \lambda(\mu(X, Y)) : (X, Y) \in \mathbf{X} \} \) is also a polytope
Example: Tensor marginals

\( X = \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n \), \( G = \text{GL}_n \times \text{GL}_n \times \text{GL}_n \)

Three different ways of viewing a \( n \times n \times n \) tensor \( X \) as an \( n \times n^2 \) matrix:

\[
\begin{align*}
M_1 & \quad M_2 & \quad M_3 \\
\end{align*}
\]

\( \mu : X \rightarrow \text{Herm}_n \times \text{Herm}_n \times \text{Herm}_n \)

\( \mu(X) = \frac{1}{\sum |X_{ijk}|^2} (M_1 M_1^*, M_2 M_2^*, M_3 M_3^*) \)

Yet another polytope:

\( \Delta(X) = \{(\lambda(\mu(X)) : X \in X\} \)

\( \mu(A), \text{ AKA partial trace of } |X\rangle \langle X| \text{ onto } i^{th} \text{ subsystem} \)
Quantum marginals

If Alice, Bob, and Carol’s qubits are jointly in a pure quantum state $X$, the one-body marginals are the $2 \times 2$ PSD matrices $\mu(X)_1, \mu(X)_2, \mu(X)_3$

![Diagram of joint state and marginals]

**One body quantum marginal problem, $d = 3$**

**Input:** PSD matrices $P_1, P_2, P_3$

**Output:** Whether there is a pure state with marginals $P_1, P_2, P_3$

**Equivalently:** whether $(\lambda(P_1), \lambda(P_2), \lambda(P_3)) \in \Delta(\text{all 3-tensors})$?
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$$\begin{array}{c}
\text{joint state } X \\
\downarrow \\
\mu(X)_1 \\
\downarrow \\
\mu(X)_2 \\
\downarrow \\
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\end{array}$$

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The solution of the one-body $n$-representability problem for three fermions with local dimension six, as given by the Borland–Dennis inequalities (3.1). The vertex $(1, 1, 1)$ corresponds to a single Slater determinant.

The solution of the one-body quantum marginal problem for pure states of three qubits, as given by the polygonal inequalities (3.2) for $n = 3$.

Bravyi’s polytope, corresponding to his solution (3.3) of the one-body quantum marginal problem for two qubits and global spectrum $AB = (0.6, 0.3, 0.1, 0)$.

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Quantum marginals

If Alice, Bob, and Carol's qubits are jointly in a pure quantum state $X$, the one-body marginals are the $2 \times 2$ PSD matrices $\mu(X)_1, \mu(X)_2, \mu(X)_3$

![Joint state diagram](image)

One body quantum marginal problem, $d = 3$

**Input:** PSD matrices $P_1, P_2, P_3$

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Equivalently: whether $(\lambda(P_1), \lambda(P_2), \lambda(P_3)) \in \Delta(\text{all 3-tensors})$?
Somewhat unifying definition

The previous moment maps are all \textit{gradients of the norm in exponential coordinates}

\[ \mu : X \mapsto \nabla_{A=0} \log \| e^A \cdot X \| \]

If \( G = \text{GL}_{n_1} \times \cdots \times \text{GL}_{n_d} \), moment polytope

\[ \Delta(X) = \{ \lambda(\mu(X)) : X \in X \} \]

lives in \( \Delta(n) := \Delta(n_1) \times \cdots \times \Delta(n_d) \), where
\[ \Delta(m) \subset \mathbb{R}^m := \text{nonnegative, decreasing vectors with unit sum}. \]

\[ \text{[Ness, Mumford '84, Kirwan '84]} \] (In differing levels of generality):
\( \Delta(X) \) is a convex polytope.
Why are moment polytopes interesting? Encode representation theory of polynomials on $\mathbb{X}$:

$R_k(\mathbb{X})$ space of degree $k$ homogeneous polynomials on $\mathbb{X}$;

$V_{G,\lambda}$ an irreducible representation of type $\lambda$

**Theorem** *(Mumford, Ness ’84, Brion ’87)*

$$\Delta(\mathbb{X}) \cap \mathbb{Q} = \{\lambda/k : V_{G,\lambda} \subset R_k(\mathbb{X})\}$$

GCT approach to VNP vs VP:

plans to prove rep thry of manifolds $R_k(\mathbb{X}_{perm})$ and $R_k(\mathbb{X}_{det})$ for specific $k$,

conjectures that this can be done by efficiently constructing $\lambda$ that can be efficiently verified to appear more times in one than the other.

Moment polytope membership is morally easier than this.
History, our results
Problem

Input: Given $\varepsilon > 0$, $p \in \Delta(n)$, access to $X$
(say by arithmetic circuit from $\mathbb{C}^r$ to $X$).

Output: $X \in X$ with

$$\|\lambda(\mu(X)) - p\|_1 \leq \varepsilon,$$

or correctly state that $p \notin \Delta(X)$.

Answers membership, but is allowed to err within $\varepsilon$ of boundary of $\Delta(X)$.
For example, if $X = G \cdot X$, input can just be $X$. 
Approximate scaling algorithms: nonuniform vs uniform

All runtimes are assumed to be polynomial in size of input

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One body quantum marginal problem ($X = \text{all } d\text{-tensors}$)
- [Higuchi, Sudbery‘02] qubits; [Klaychko‘04] polytope, [WDGC13] algebraic algorithm, [BCMW’17]: membership problem in $\text{NP} \cap \text{coNP}$
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- $[BFGOWW \ '18]$ poly($1/\varepsilon$)
Our results

**Theorem (BFGOWW ’18, Tensor scaling)**

There is a randomized $\text{poly}(n^d, \langle X \rangle + \langle p \rangle, 1/\varepsilon)$-time algorithm for $X = G \cdot (\geq d\text{-tensor})$ on input $X, p, \varepsilon$ with success probability $1/2$.

The algorithm requires $O\left( dn^{2d} \frac{\langle X \rangle + \langle p \rangle + d \log dn}{\varepsilon^2} \right)$ iterations, each dominated by computing a Cholesky decomposition of some $n \times n$ matrix.

**Corollary (BFGOWW ’18, Quantum marginals)**

There is a randomized $\text{poly}(\langle p \rangle, 1/\varepsilon)$-time algorithm for $X = \text{all } d\text{-tensors}$ on input $p, \varepsilon$ with success probability $1/2$.

**Corollary (BFGOWW ’18, Approximate moment polytope membership)**

There is a randomized algorithm for approximate moment polytope membership running in time $\text{poly}(n^d, \langle \text{parameterization of } X \rangle, p, 1/\varepsilon)$.
Theorem (BFGOWW ‘18)

If $\|\lambda(\mu(X)) - p\|_1 \leq \exp(-O(dn^{d+1})\langle p \rangle)$, then $p \in \Delta(X)$.

Unfortunately, doesn’t result in poly time algorithm for membership! Need $\text{poly}(\log(1/\varepsilon))$ run time.
Algorithms
Optimization

First solve the problem when $X = \Delta(G \cdot X)$, then solve for random case. Easiest to describe when $p \propto (p_1 l_1, \ldots, p_n l_d)$. In this case,

**Theorem (Mumford, Ness ’84)**

\[
p \in \Delta(G \cdot X) \iff \inf_{g \in G} \frac{\|g \cdot X\|}{\prod_i |\det g_i|^{p_i}} > 0.
\]

**Proof sketch:**

$\Leftarrow$ log of r.h.s. will achieve gradient zero even in exponential coordinates;

$\nabla_{A=0}(\log \|e^A \cdot X\| - \sum p_i \log |\det e^{A_i}|) = 0$ implies $\mu(X) = (p_1 l_1, \ldots, p_n l_d)$.

$\Rightarrow$ Showing that a gradient condition *somewhere* implies $\text{opt} > 0$...need some type of convexity.
Geodesic convexity

Function we want, \( f_{p,X} := \log \frac{\|g \cdot X\|}{\prod_i |\det g_i|^{\rho_i}} \), is convex in every direction:

\[ f_{p,X}(e^{At}) \text{ is a convex function in } t \]

is convex in \( t \).
Nonuniform, focus on $d = 2$

$p = (p_1, p_2)$ not each proportional to $1$:
$f_{p,\chi}(g)$ gets crazier:

$$f_{p,\chi}(g) = \log \frac{\|g \cdot X\|}{\chi_p(g)} := \log \frac{\|g \cdot X\|}{|\chi_{p_1}(g_1)\chi_{p_2}(g_2)|}$$

$\chi$ makes sense on lower triangular matrices $b$: $\chi_q(b) := \prod g_{ii}^{q_i}$

**Theorem (Mumford, Ness ’84)**

$p \in \Delta(G \cdot X) \iff$ for $Y = g_0 \cdot X$

$$\text{cap}(Y) := \inf_{g \text{ lower triangular}} f_{p, Y}(g) > 0$$

with probability 1 over random choice of $g_0$. 
• **Coordinate descent:** Matrices: Sinkhorn scaling [Si64], Tensors [F18], [BGOWW17], [BFGOWW18], gives poly(1/ε) runtime.

• **Trust region methods:** [AGLOW17] operator scaling, [BFGOWW18+] for general setting, gives

\[
\text{poly}(1/\gamma, \log(1/\varepsilon))
\]

runtime.

What’s γ? Essentially minimum distance Δ(X) can be from \( p \) without containing \( p \).

• For \( d = 2 \), uniform case, \( \gamma \geq 1/\text{poly}(n) \),

• for \( d = 3 \), \( \gamma = \exp(-n) \).
Analysis
Capacity lower bounds

All the algorithms are implicitly poly(cap), so in general we have to show that cap is bounded below.
Our lower bounds are objects arising from representation theory

**Definition** \((\text{HWV}_p^k)\)

degree \(k\) homogeneous polynomials that are eigenfunctions for the action of the lower triangular matrices;

\[
P(g \cdot Y) = \chi_p(g)^k P(Y)
\]

There’s a fairly easy *weak duality* here: very roughly,

\[
\inf_{g} f_{p,Y}(g) \geq \sup_{P \in \text{HWV}_p^k} \frac{1}{k} \log \frac{P(Y)}{\|P\|}
\]

Norm here is *Bombieri norm* on polynomials
$p \in \Delta(\mathcal{G} \cdot X)$
Moment polytope and representation theory

\( p \in \Delta(\mathcal{G} \cdot \mathcal{X}) \)

\( \lambda_{1,1} \)

\( \lambda_{0,1} \)

\( \lambda_{1,0} \)

\( \lambda_{0,0} \)

\( 1 \)

\( 0.5 \)

\( 0 \)

\( \inf_g f_{p, \gamma}(g) > \frac{1}{\text{poly}} \)

Want for

\( Y = g_0 \cdot X \)

random

force

\( \text{potential} \)

Ness - Mumford '84 + Derksen '01

Want for

\( Y = g_0 \cdot X \)

random

Probability > 0.5
Moment polytope and representation theory

\[ p \in \Delta(\overline{G \cdot X}) \]

\[ Q \neq 0 \text{ on } \overline{G \cdot X}, \ Q \in HWV_p^k \]
\[ \deg Q \leq 2^{\text{poly}} \]

Want for
\[ Y = g_0 \cdot X \]
\[ \text{random} \]

\[ \inf_g f_{p, Y}(g) > \frac{1}{\text{poly}} \]

\[ \text{Ness–Mumford'84} \]
\[ + \text{Derksen'01} \]
Moment polytope and representation theory

\[ p \in \Delta(G \cdot X) \]

\[ Q \not\equiv 0 \text{ on } \overline{G \cdot X}, \quad Q \in \text{HWV}_p^k \]
\[ \deg Q \leq 2^{\text{poly}} \]

Want for
\[ Y = g_0 \cdot X \]
random

\[ \inf_g f_{p,Y}(g) > \frac{1}{\text{poly}} \]

\[ \text{probability} > .5 \]

\[ Q(Y) \not\equiv 0 \]
Moment polytope and representation theory

\[ p \in \Delta(G \cdot X) \]

\[ Q \neq 0 \text{ on } G \cdot X, \quad Q \in HWV^k_p \]
\[ \text{deg } Q \leq 2^{\text{poly}} \]

\[ Q(Y) \neq 0 \]

Want for
\[ Y = g_0 \cdot X \]
random

\[ \inf_g f_{p,Y}(g) > \frac{1}{\text{poly}} \]

\[ \sup_{P \in HWV^k_p} \frac{1}{k} \log \frac{|P(Y)|}{\|P\|} > \frac{1}{\text{poly}} \]
Moment polytope and representation theory

\[ p \in \Delta(G \cdot X) \]

\[ Q \not\equiv 0 \text{ on } G \cdot X, \quad Q \in HWV_p^k \]

\[ \deg Q \leq 2^{\text{poly}} \]

\[ \inf_g f_{p,Y}(g) > \frac{1}{\text{poly}} \]

\[ \sup_{P \in HWV_p^k} \frac{1}{k} \log \frac{|P(Y)|}{\|P\|} > \frac{1}{\text{poly}} \]
Open problems

• Is the tensor scaling decision problem in \( \text{NP} \)? Is it in \( \text{coNP} \)?
• Is it in \( \text{RP} \)? A poly log\((1/\varepsilon)\) algorithm would prove it! In \( \text{P} \)?
• Can tensor scaling be done in poly log\((1/\varepsilon)\) for a random tensor? Would put quantum marginal problem in \( \text{RP} \)!
• Obtain similar algorithms for \textit{multi-body} quantum marginals.
• Develop separation oracles for moment polytopes.
Thank you!