Motivation, Definitions and Notation: Recall that the proof of Gödel’s Incompleteness Theorem involved the construction of a formula $\phi_{M,x}$ of the form $\phi_{M,x} = \neg \exists v \text{VALCOMP}_{M,x}(v)$ with the property that $\mathbb{N} \models \phi_{M,x}$ if and only if Turing machine $M$ does not halt on input $x$. Let $\text{PA}$ denote Peano Arithmetic. The Incompleteness Theorem says that there is a Turing machine $M$ and an input $x$ such that $\mathbb{N} \models \phi_{M,x}$, but such that there is no proof in $\text{PA}$ of the logic formula $\phi_{M,x}$. That is, the machine $M$ really does not halt on input $x$, but this cannot be proved in $\text{PA}$. For the remainder of this write-up, we will use the notation “$M$” and “$x$” to refer to this specific machine $M$, which does not halt on input $x$, but where this cannot be proved in $\text{PA}$.

Let $\Gamma$ be any set of formulae. Recall that the notation $\Gamma \models \phi$ means that, for every structure $M$ such that $M \models \Gamma$, it also holds that $M \models \phi$. In particular, if there is no structure that satisfies $\Gamma$, then the condition “$M \models \phi$” is considered to hold vacuously (since it holds for “every structure that satisfies $\Gamma$”), and thus, in this case we say that $\Gamma \models \phi$ for every formula $\phi$, including obvious contradictions such as the case when $\phi$ is $\psi \land \neg \psi$. In this case, we say that $\Gamma$ is inconsistent.

The notation $\Gamma \vdash \phi$ means that there is a proof (using some standard notion of “proof system”) such that every line of the proof is either an element of $\Gamma$ or else follows from some earlier lines according to the inference rules of the proof system, where the final line in the proof is $\phi$.

Gödel’s Completeness Theorem says that $\Gamma \models \phi$ implies $\Gamma \vdash \phi$.

In particular, the proof of the Completeness Theorem establishes that, if $\Gamma$ is inconsistent, then $\Gamma \vdash (\psi \land \neg \psi)$. This proof must be of finite length, and thus it can only make use of finitely-many of the formulae in $\Gamma$. This proves:

The Compactness Theorem for first-order logic: If $\Gamma$ is inconsistent, then there must be a finite subset $\Gamma' \subseteq \Gamma$ that is inconsistent (because $\Gamma' \vdash (\psi \land \neg \psi)$).

For the particular case where $\Gamma = \text{PA}$, combining the completeness theorem and the incompleteness theorem, we have that $\text{PA} \nvdash \phi_{M,x}$, and thus $\text{PA} \nmodels \phi_{M,x}$ — and thus (by definition) there must be a structure $M$ such that $M \models \text{PA}$ and simultaneously $M \models \neg \phi_{M,x}$. That is, $M$ “looks like” $\mathbb{N}$ (in the sense that it satisfies $\text{PA}$, which are the usual axioms for $\mathbb{N}$), but nonetheless $M \models \exists v \text{VALCOMP}_{M,x}(v)$. That is, in the structure $M$, there is a “number” that encodes a halting computation transcript of $M$ on input $x$. This homework assignment asks you to explore how this can possibly be.
1. Let $\Gamma_0$ be PA. Let $c$ be a new constant. Consider $\Gamma_1 = \Gamma_0 \cup \{c > 1, c > 1 + 1, c > 1 + 1 + 1, \ldots \}$. Prove that every finite subset of $\Gamma_1$ is consistent. Then explain why this shows that there is a structure $\mathcal{M}$ such that $\mathcal{M} \models \Gamma_1$.

$\mathcal{M}$ is called a non-standard model of arithmetic. Note that $\mathcal{M}$ contains elements that are larger than any “standard” integer.

In the rest of this assignment, fix one such “non-standard” structure $\mathcal{M}$.

2. PA proves several standard facts about $\mathbb{N}$, such as

$$\forall x \ (x > 0 \rightarrow (\exists y \ (y + 1 = x \land y \neq x)))$$

and $\forall x \forall y (x < y \lor y < x \lor y = x)$, and $\forall x \exists y (y^2 \leq x \land (y + 1)^2 > x)$. (This third statement can be interpreted as saying “for all $x$, $\lceil \sqrt{x} \rceil$ exists”.) What does this allow you to say about the interpretation of the constant $c$ in $\mathcal{M}$? Do the elements $c - 1$ and $\lceil (c)^{1/2} \rceil$ exist? If so, are these elements standard or non-standard elements? Is it correct to think of $\mathcal{M}$ as being $\mathbb{N} \cup \{\infty\}$, or is it something more complicated?

3. Consider the closed terms that do not use the constant $c$ (such as $(1 + 1) \times (1 + 1 + 1)$). Do these terms represent standard or non-standard elements of $\mathcal{M}$?

4. Now consider any structure $\mathcal{M}'$ such that $\mathcal{M}' \models \exists v \ \text{VALCOMP}_{\mathcal{M},x}(v)$ and $\mathcal{M}' \models \text{PA}$. (That is, we are not necessarily assuming ahead of time that $\mathcal{M}'$ satisfies $\Gamma_1$ or even that $\mathcal{M}'$ has an interpretation of the constant $c$.) Show that we can nonetheless conclude that $\mathcal{M}'$ contains nonstandard elements. (Hint: consider the formulas of the form $\text{VALCOMP}_{\mathcal{M},x}(1 + 1 + 1 + 1)$, and with other closed terms plugged in place of the variable $v$.)