1. Let $A$ and $B$ be languages. We say that $A \leq_m B$ if there is a computable function $f$ such that, for every string $x$:

$$x \in A \Rightarrow f(x) \in B$$

and

$$x \notin A \Rightarrow f(x) \notin B.$$  

(This is a very important relation on languages.)

Show that $\leq_m$ is a transitive relation. That is, show that if $A \leq_m B$ and $B \leq_m C$, then $A \leq_m C$. If $A \leq_m B$ and $B$ is recursive, is $A$ recursive? If $B$ is r.e., is $A$ recursively-enumerable?

If $A \leq_m B$ and $A$ is recursive, is $B$ recursive? If $A$ is r.e., is $B$ recursively-enumerable? Justify your answers.

2. Explain why $\leq_m$ is not a partial order. (As part of this exercise, you might need to do some searching, to find out what a partial order is.)

3. Let $A$ and $B$ be languages. We say that $A \equiv_m B$ if $A \leq_m B$ and $B \leq_m A$. Show that $\equiv_m$ is an equivalence relation.

4. Let $HP$ denote the halting problem. Consider the language:

$$A = \{1x : x \in HP\} \cup \{0x : x \notin HP\}.$$  

Show $HP \leq_m A$ and $\overline{HP} \leq_m A$. Is $A$ recursively-enumerable? Justify your answer.

5. Show that $A$ is decidable if and only if $A$ can be recursively enumerated in lexicographic order.