# ***Tree Packing in Complete Graph***

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**Abstract**

A conjecture of Gyárfás and Lehel asks if the sequence of trees $T_1, T_2, ..., T_n$, where $T_i$ is a tree on $i$ vertices, can be packed into $K_n$. We show that if each $T_i$ is restricted to a star or a path, then the sequence can be packed. We also give an explicit construction for a restricted case when the paths and stars alternate.

## 1. Introduction

We say that a sequence of graphs $G_1, G_2, ..., G_k$ is packed into the graph $G$, if there exist edge-disjoint subgraphs $H_1, H_2, ..., H_k$ of $G$ such $H_i$ is isomorphic to $G_i$. The Tree Packing conjecture of Gyárfás and Lehel, (1974) asks if the set of $n$ trees, $T_1, T_2, ..., T_n$, where $T_i$ is on $i$ vertices can be packed exactly into $K_n$; where $K_n$ is a complete graph on $n$ vertices. This problem has got applications in Network Coding solutions for information multicast. A. Gyárfás and J. Lehel in [1] proved that $T_1, T_2, ..., T_n$ can be packed in $K_n$ provided that all but two of them are stars. If each tree is a path then by a result of Fink and Straight[3], they can be packed into $K_n$.

Several related problems have drawn substantial research attention. A similar conjecture of Hobbs, Bourgeois and Kasiraj [7] asks if the sequence of trees can be packed into complete bipartite graphs. Both the conjectures are still open and most of the results are partial packing guarantees, in [5] it is proved that $T_1, T_2, ..., T_{\frac{n}{\sqrt{2}}}$ can be packed into $K_n$, a bipartite analog of this in [6] shows that $T_1, T_2, ..., T_{\sqrt{\frac{5}{8}}n}$ can be packed into $K_{n-1,\frac{n}{2}}$.

Our result is for a restricted version of the conjecture, where each $T_i$ is restricted to be either a star or a path on $i$ vertices. We denote a star and path on $i$ vertices by $S_i$ and $P_i$. A star on $n$ vertices, $S_n$, is a tree with one vertex having degree $n$ and the others having degree 1. A path on $n$ vertices, $P_n$, is a tree with two vertices of degree 1, and the other $n-2$ vertices of degree 2.

In section 2 we give an explicit construction for the sequence of tree in which paths and stars alternate, in section 3 we prove that any sequence of paths and stars can be packed. The result in section 2 is a special case of that in section 3 but the proof is different and we believe that the technique can be applied to get other packing results. The theorem in section 3 is already proven in [2], but we give a graph theoretic version of the proof.

## 2. Packing alternating sequence of paths and stars

We further simplify the problem by imposing the condition on the sequence of trees to be alternating between stars and paths starting with a path, as the initial star can be omitted by packing the remaining sequence into $K_{n-1}$. So the problem becomes of packing $P_n, S_{n-1}, P_{n-2}, S_{n-3}, ....., P_1$ in $K_n$.

We will give detailed construction for the case when n is even, for n odd the result simply follows with slight modification. Let the vertices of complete graph $K_n$ be $v_0, v_1, ....., v_{2k-1}$. Arrange these vertices in a regular $n$-gon in counter-clockwise direction and join every two vertices with a line segment.

Define a zigzag path of length $k$ starting at $v_i$, $Z(v_i, k) = \hat{e}_0, \hat{e}_1, ....., \hat{e}_{k-1}$ such that $\hat{e}_0 = v_i v_{i+1}$, $\hat{e}_1 = v_{i+1} v_{i-1}$ and for each $\hat{e}_j = v_m v_n, (j < k-1)$ $\hat{e}_{j+1} = v_n v_l$ for some $l$, such that $\hat{e}_{j+1}$ is parallel to $\hat{e}_{j-1}$, where the subscripts of vertices are expressed modulo $n$. Now map each path $P_k$ of the sequence to the zigzag path $Z(v_{\frac{n+k}{2}}, k-1)$. Note that the successive zigzag paths are edge disjoint.

This successive zigzag paths are illustrated in following figures.

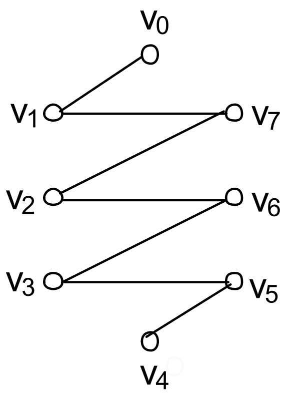


Figure 1: $Z(v_0, 7)$

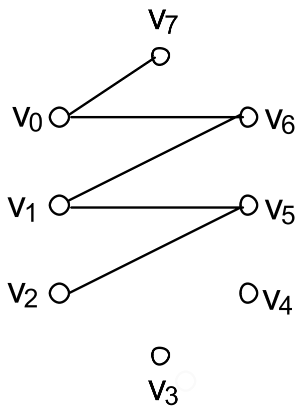


Figure 2: $Z(v_7, 5)$

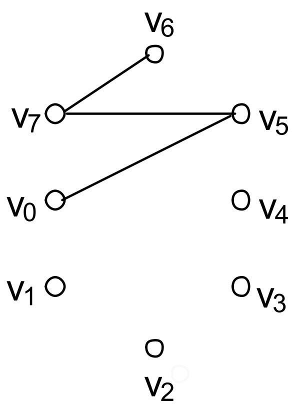


Figure 3: $Z(v_6, 3)$

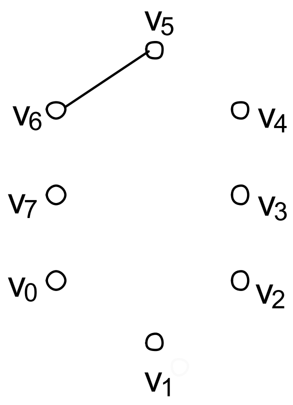


Figure 4: $Z(v_5, 1)$

**Observation 1**: In the above procedure, exactly one edge incident to $v_{\frac{n}{2}}$ is mapped to some edge and deleted; hence in the remaining graph the degree of $v_{\frac{n}{2}}$ is $n-2$

**Observation 2:** All the edges incident to $v_{\frac{n}{2}+1}$ are mapped to and deleted; hence in the remaining graph the degree of $v_{\frac{n}{2}+1}$ is 0.

**Observation 3:** It is not hard to see that exactly $\frac{n}{2}$ edges incident to $v_0$ are deleted.

**Lemma 1:** Let $d(v_i)$ be the degree of $v_i$ in the remaining graph then $d(v_0) = \frac{n}{2} - 1$ and $d(v_i) = (i - \frac{n}{2} - 1) \bmod (n-1)$ for $(1 \leq i \leq n-1)$

What remains to be packed is a sequence of stars $S_{n-1}, S_{n-3}, ...S_1$. By Lemma 1, we have $d(v_{\frac{n}{2}}) = n-2$ and $d(v_{\frac{n}{2}}+1) = 0$, and the degree of center of $S_{n-1}$ is $n-2$. Furthermore, $v_{\frac{n}{2}}$ is connected to all vertices but $v_{\frac{n}{2}+1}$. So we will map edges of $S_{n-1}$ to the edges incident to $v_{\frac{n}{2}}$. By deleting all these edges, the degree of each vertex other than $v_{\frac{n}{2}}$ is reduced by one and that of $v_{\frac{n}{2}}$ becomes zero. Hence we introduce two new zero degree vertex. Map edges of $S_{n-3}$ to the edges incident to $v_{\frac{n}{2}-1}$. In general map edges of $S_{n-(2i+1)}$ to edges incident to $v_{\frac{n}{2}-i}$

**The Odd Case:** Arrange the vertices $v_1, v_2, ....., v_{2k}$ of complete graph $K_n$ in a regular $2k$-gon in counter-clockwise direction and place $v_0$ at the top. Now the define the odd zigzag path, $\hat{Z}(v_i, k)$ to be $\{v_0 v_i\} \cup Z(v_i, k-1)$. Now map each path $P_k$ of the sequence to the zigzag path $\hat{Z}(v_{\frac{n+k}{2}+1}, k-1)$. The rest of argument is exactly the same as for the even case, except that in the remaining graph after packing all the paths, $d(v_0) = \lfloor \frac{n}{2} \rfloor$

**Theorem 1:** $T_1, T_2, ..., T_n$, where $T_i = S_i$ for $i$ odd and $T_i = P_i$ for $i$ even can be packed into $K_n$.

**3. Packing Stars and Paths**

In this section we show that any sequence of paths and stars can be packed into $K_n$. We give detailed construction for the case when n is even, for n odd the result is very similar and is omitted.

**Theorem 2:** $T_1, T_2, ..., T_n$, where each $T_i$ is either $S_i$ of $P_i$, can be packed into $K_n$ for $n$ even.

**Proof**. Let the vertices of $K_n$ be $v_1, v_2, ....., v_{2k}$. Partition $K_n$ into two disjoint subgraph graphs $G_1(n)$ and $G_2(n)$ such that

$V(G_1(n)) = V(G_2(n)) = V(K_n)$ ,
$E(G_1(n)) = \{v_iv_j : 1 \leq i \leq \frac{n}{2} - 1; (i+1) \leq j \leq (n-i)\}$ and
$E(G_2(n)) = \{v_iv_j : 1 \leq i \leq n; (n-i+1) \leq j \leq n\}$

This Decomposition of $K_n$ is illustrated in the Figure 5 and 6

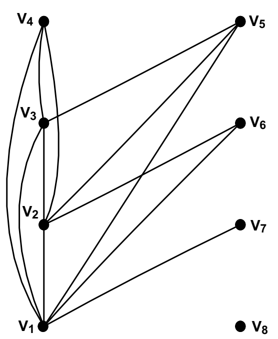

Figure 5: $G_1$

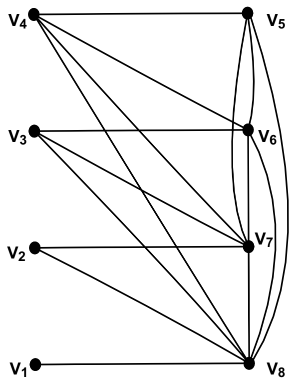

Figure 6: $G_2$

We now show that $T_i$ can be packed into $G_1(n)$ for odd $i$ and $T_i$ can be packed into $G_2(n)$ for even $i$. We only prove that all $T_i$ for odd $i$ can be packed into $G_1(n)$ by induction on $n$; while the proof for $G_2(n)$ is similar.

For $n = 2, G_1(n)$ is an empty graph on 2 vertices while $T_1$ is just a vertex, which can be packed into $G_1(n)$. Assume that it is true for an even $m < n$. Then for $n$ we want to pack $T_1, T_3, ..., T_{n-1}$ into $G_1(n)$, where each $T_i$ is either $S_i$ or $P_i$. If $T_{n-1} = S_{n-1}$ then mapping $T_{n-1}$ to $v_1$ as its center, and removing it, we are left with $G_1(n-1)$ in which we can pack all the rest of $T_i$'s by the induction hypothesis. If $T_{n-1} = P_{n-1}$ then map $T_{n-1}$ to the path $v_{k_0}v_{k_1}, v_{k_1}v_{k_2}, ....., v_{k_{n-3}}v_{k_{n-2}}$ where $k_j = \frac{n}{2} + (-1)^j\lceil\frac{j}{2}\rceil$. Removing this path we are left with $G_1(n-1)$ in which we can pack all the remaining $T_i$'s by the induction hypothesis.

The deletion of $T_{n-1}$ is shown in figure 7 to 10.

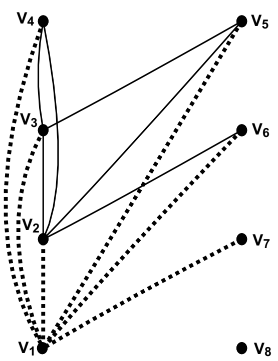


Figure 7: $S_{n-1}$ selected in $G_1(n)$

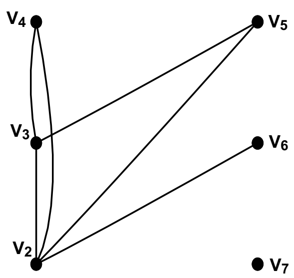


Figure 8: After deleting $G_1(n-1)$

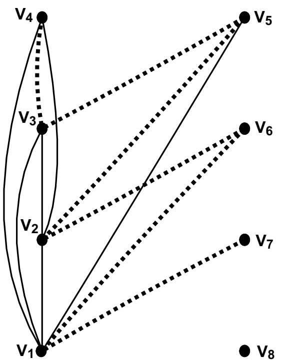


Figure 9: $P_{n-1}$ selected in $G_1(n)$

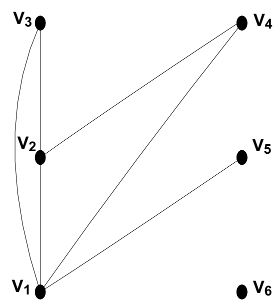


Figure 10: After deleting $G_1(n-1)$

## 4. References


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