ON THE EXISTENCE OF APPROXIMATE SOLUTIONS FOR SINGULAR INTEGRAL EQUATIONS OF CAUCHY TYPE DISCRETIZED BY GAUSS-CHEBYSHEV QUADRATURE FORMULAE

by

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DCS-TR-94

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September, 1980
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ABSTRACT

It is shown that the direct Gauss-Chebyshev method used for the numerical solution of singular integral equations of Cauchy-type possesses a unique solution for any \( \alpha \), as long as \( \lambda \) is not an eigenvalue of the equation.
1. Introduction

Erdogan [1], Erdogan and Gupta [2] have introduced two numerical methods for the solution of the Cauchy type singular integral equation,

\[ \frac{1}{\pi} \int_{-1}^{1} g(t) \frac{dt}{t-s} + \lambda \int_{-1}^{1} K(s,t) g(t) dt = f(s), \quad -1 < s < 1 \]

subject to an additional condition of the form

\[ \int_{-1}^{1} g(t) dt = \eta. \]

The index theory of such equations (see [6]), implies that \( g(t) \) is unbounded at the end points \( -1 \), and may be written as

\[ g(t) = y(t)(1-t^2)^{-1/2}, \]

where \( y(t) \) is a continuous function in \([-1,1]\).

The method in [1] consists of approximating \( y(t) \) in terms of Chebyshev polynomials, i.e.

\[ y(t) = y_n(t) = \sum_{i=0}^{n} a_i T_i(t) \]

where \( T_i(x) \) is the Chebyshev polynomial of order \( i \), and determining the unknown coefficients \( a_i \) using orthogonality identities of the Chebyshev polynomials.
Convergence of the approximate solution \( y_n(t) \) towards \( y(x) \) was shown by Linz in [3]. Linz also found that the order of convergence of the method is \( O(n^{-p}) \), if \( f(s) \) is continuously differentiable \( p+1 \) times, \( 1 < p \).

In [2] Erdogan and Gupta show that the well known Gauss-Chebshev quadrature formula can be applied on a certain set of points to approximate (1.1). They approximate \( y(x) \) with

\[
y_n(t) = \sum_{i=0}^{n} a_i T_i(t),
\]

and show that,

\[
\int_{-1}^{1} (1-t^2)^{-1/2} y(t) dt = \frac{\pi}{n+1} \sum_{i=0}^{n} \frac{y_n(t_i)}{t_i - s_r},
\]

If

\[
T_n+1(t_i) = 0, \quad i=0, \ldots, n;
\]

\[
U_n(s_r) = 0, \quad r=0, \ldots, n-1,
\]

where \( T_n, U_n \) are the Chebyshev polynomials of the first and second kind respectively. After introducing (1.4) into (1.1) a linear algebraic system with unknowns \( y_n(t_i), i=0, \ldots, n \) is obtained.

Ioakimidis and Theocaris [4] assume that the algebraic system has a solution, \( y_n(t_i), i=0, \ldots, n \) and construct an interpolating polynomial \( y_n(t) \) using these values. Afterwards they show that Linze's error analysis can be extended to the Gauss-Chebshev method.
It is obvious that their conclusions will be true only if the algebraic system has a unique solution for any \( n \). But there is no mathematical reason to believe that the linear algebraic system possesses such a solution.

Srivastav in [5] uses identities involving the zeros of Chebyshev polynomials, to obtain a closed form expression of the inverse of the coefficient matrix, in the case that \( \lambda = 0 \) in (1.1).

In Section 2, a brief description of the Gauss-Chebyshev method, and closed form expressions of the coefficient matrix and its inverse for \( \lambda = 0 \) are given. Also a very simple form of the square of the determinant is obtained. We use this form to show that the coefficient matrix is nonsingular for sufficiently large \( n \).

From the general theory of singular integral equation [6], we know that equation (1.1) is equivalent to a Fredholm Integral Equation, which may be solved by using regularization techniques.

In Section 3 a method based on the approximation of the reduced Fredholm Integral Equation is given. It is shown that both direct Gauss-Chebyshev [2] and the proposed method are equivalent. This in turn implies that coefficient matrix of the direct Gauss-Chebyshev method is nonsingular for any \( n \), as along as \( \lambda \) is not an eigenvalue of (1.1).
2. The coefficient matrix in the case of $\lambda = 0$

Let us introduce (1.4) in (1.1), and approximate the second part of (1.4) using the Gauss-Chebyshev quadrature formula. Then (1.1), (1.2) becomes

\[
\begin{align*}
\frac{1}{m} \sum_{j=1}^{m} y_m(t_j) + \lambda \sum_{j=1}^{m} K(s_k, t_j) y_m(t_j) &= f(s_k), \\
\sum_{j=1}^{m} y_m(t_j) &= 0,
\end{align*}
\]

(2.1)

where $t_j = \cos[(2j-1)/2m]$, $j = 1, \ldots, m$

$s_k = \cos[k/m]$, $k = 1, \ldots, m-1$.

Following [5] we assume for simplicity that $\lambda = 0$, and $f(s)$ is even. Then (2.1) is reduced to

\[
\begin{align*}
\frac{2}{2n+1} \sum_{j=1}^{n} \frac{y_n(t_j) t_j}{t_j^2 - s_k^2} &= f(s_k),
\end{align*}
\]

(2.2)

and

\[
\begin{align*}
t_j &= \cos[(2j-1)\pi/2(2n+1)], \quad j = 1, \ldots, n \\
s_k &= \cos[k\pi/(2n+1)], \quad k = 1, \ldots, n.
\end{align*}
\]

(2.3)

Equation (2.2) represents an algebraic system of linear equations, with unknowns $y_n(t_j)$, $j = 1, \ldots, n$. It may be rewritten as a matrix equation.
\[ (2.4) \quad \mathbf{A}_n \mathbf{y} = \mathbf{f} \]

where

\[
\begin{pmatrix}
  \frac{t_1}{2} & \frac{t_2}{2} & \cdots & \frac{t_n}{2} \\
  \frac{t_1 - s_1}{2} & \frac{t_2 - s_2}{2} & \cdots & \frac{t_n - s_n}{2} \\
  \frac{t_1 - s_1}{2} & \frac{t_2 - s_2}{2} & \cdots & \frac{t_n - s_n}{2} \\
  \vdots & \vdots & \ddots & \vdots \\
  \frac{t_1 - s_1}{2} & \frac{t_2 - s_2}{2} & \cdots & \frac{t_n - s_n}{2} \\
\end{pmatrix}
\]

\[ (2.5) \quad \mathbf{A}_n = \frac{2}{2n+1} \begin{pmatrix}
  \frac{t_1}{2} & \frac{t_1}{2} & \cdots & \frac{t_1}{2} \\
  \frac{t_1 - s_1}{2} & \frac{t_2 - s_2}{2} & \cdots & \frac{t_n - s_n}{2} \\
  \frac{t_1 - s_1}{2} & \frac{t_2 - s_2}{2} & \cdots & \frac{t_n - s_n}{2} \\
  \vdots & \vdots & \ddots & \vdots \\
  \frac{t_1 - s_1}{2} & \frac{t_2 - s_2}{2} & \cdots & \frac{t_n - s_n}{2} \\
\end{pmatrix}
\]

and \( \mathbf{y} = [y_1(t_1), \ldots, y_n(t_n)]^T, \mathbf{f} = [f(s_1), \ldots, f(s_n)]^T \).

The inverse \( \mathbf{A}_n^{-1} \) of the matrix \( \mathbf{A}_n \) is given by

\[ (2.6) \quad \mathbf{A}_n^{-1} = \frac{2}{2n+1} \begin{pmatrix}
  \frac{t_1(1-s_1^2)}{2} & \frac{t_1(1-s_1^2)}{2} & \cdots & \frac{t_1(1-s_1^2)}{2} \\
  \frac{t_1 - s_1}{2} & \frac{t_2 - s_2}{2} & \cdots & \frac{t_n - s_n}{2} \\
  \frac{t_1 - s_1}{2} & \frac{t_2 - s_2}{2} & \cdots & \frac{t_n - s_n}{2} \\
  \vdots & \vdots & \ddots & \vdots \\
  \frac{t_1 - s_1}{2} & \frac{t_2 - s_2}{2} & \cdots & \frac{t_n - s_n}{2} \\
\end{pmatrix}
\]

(for more details see [5]).
We are now ready to show that the coefficient matrix $A_n$ is nonsingular for all $n$.

**Theorem 1.** The square of the determinant of the matrix $A_n$ is greater than 1 for any integer $n$.

**Proof.** It is easily seen from (2.5) and (2.6) that

$$
\det A_n^{-1} = \det A_n \prod_{j=1}^{n} (1-s_j^2).
$$

Using the identity $\det A_n^{-1} \det A_n = 1$, the previous equation becomes

$$
(\det A_n)^2 = \frac{1}{\prod_{j=1}^{n} (1-s_j^2)}.
$$

From (2.3) we have that $1-s_j^2 = \sin^2(\frac{j\pi}{2(n+1)})$

and $\prod_{j=1}^{n} (1-s_j^2) < 1$ for any integer $n$, which in turn implies

that

$$
(\det A_n)^2 > 1.
$$

The last relation shows that $|\det A_n| > 1$ independently of $n$, which means that $\det A_n \neq 0$ for any $n$.

If instead we assume that $\lambda = 0$ and $f(s)$ is odd, then the matrix equation becomes (see [5]),
\[(2.10) \quad \hat{y}_n = f \]

where

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
\frac{t_1^2 - s_1}{n} & \frac{t_2^2 - s_2}{n} & \cdots & \frac{t_n^2 - s_n}{n} \\
1 & \frac{t_1^2 - s_1}{n} & \cdots & \frac{t_n^2 - s_n}{n} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \frac{t_1^2 - s_1}{n} & \cdots & \frac{t_n^2 - s_n}{n} \\
1 & 1 & \cdots & 1 \\
\end{bmatrix}
\]

\[(2.11) \quad \bar{a}_n = \frac{2}{n} \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\frac{t_1^2 - s_1}{n} & \frac{t_2^2 - s_2}{n} & \cdots & \frac{t_n^2 - s_n}{n} \\
\cdots & \cdots & \ddots & \cdots \\
1 & \frac{t_1^2 - s_1}{n} & \cdots & \frac{t_n^2 - s_n}{n} \\
\end{bmatrix}
\]

\[y = [y_n(t_1), \ldots, y_n(t_n)]^T, \quad f = [f(s_1), \ldots, f(s_{n-1}), N]^T,\]

and

\[t_j = \cos((2j-1)\pi/4n), \quad j = 1, \ldots, n\]

\[s_k = \cos(k\pi/2n), \quad k = 1, \ldots, n-1\]

The inverse of the matrix \( \bar{a}_n \) is

\[(2.12) \quad \bar{a}_n^{-1} = \frac{2}{n} \begin{bmatrix}
\frac{s_2^2(1-s_2^2)}{t_2^2 - s_2^2} & \frac{s_2^2(1-s_2^2)}{t_2^2 - s_2^2} & \cdots & \frac{s_2^2(1-s_2^2)}{t_2^2 - s_2^2} \\
\frac{s_1^2(1-s_1^2)}{t_1^2 - s_1^2} & \frac{s_2^2(1-s_2^2)}{t_2^2 - s_2^2} & \cdots & \frac{s_2^2(1-s_2^2)}{t_2^2 - s_2^2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{s_1^2(1-s_1^2)}{t_1^2 - s_1^2} & \frac{s_2^2(1-s_2^2)}{t_2^2 - s_2^2} & \cdots & \frac{s_2^2(1-s_2^2)}{t_2^2 - s_2^2} \\
\end{bmatrix}
\]
Theorem 2. The square of the determinant of $B_n$ is greater than 1 for any integer $n$.

Proof. Following similar arguments as in Theorem 1 we can easily see that

$$(\det B_n)^2 = \frac{1}{\prod_{j=1}^{n-1} s_j^2 (1 - s_j^2)}$$

(2.13)

which implies that $\det B_n \not= 0$ if $n$ is finite.

To see that $\det B_n \not= 0$ for any $n$, we notice that

$$\prod_{j=1}^{n-1} s_j^2 (1 - s_j^2) = 2^{2(n-1)} \prod_{j=1}^{n-1} \sin^2 \left( \frac{\pi j}{n} \right)$$

and that

$$(\det B_n)^2 = \frac{2^{2(n-1)}}{n \prod_{j=1}^{n-1} \sin^2 \left( \frac{\pi j}{n} \right)} > 1.$$

(2.14)

In general, if $f(s)$ is neither odd or even we may still show the existence of a solution of the approximate equation (2.1) with $\lambda = 0$. To do that we replace in (2.1), $s_k$ with $-s_k$ and $t_j$ with $-t_j$, and obtain the equation

$$\sum_{j=1}^{m} \frac{y_m(-t_j)}{t_j - s_k} = f(-s_k)$$

(2.15)
Equations (2.1) and (2.15) are equivalent to

\[ \frac{1}{m} \sum_{j=1}^{m} \frac{y_m(t_j) + y_m(-t_j)}{t_j - s_k} = f(s_k) + f(-s_k) \]

(2.16)

\[ \frac{1}{m} \sum_{j=1}^{m} \frac{y_m(t_j) + y_m(-t_j)}{t_j - s_k} = f(s_k) + f(-s_k) \]

where now \( f(s) + f(-s) \) and \( f(s) - f(-s) \) are even and odd functions respectively.

Finally as a conclusion of this section we may say that since the existence of unique solution of (2.15) is assured from the previous analysis, the coefficient matrix of (2.4) is non-singular for any function \( f(s) \) and \( \lambda \neq 0 \).

3. The coefficient matrix in the case of \( \lambda \neq 0 \)

In this section we will show that the coefficient matrix is nonsingular for \( \lambda \neq 0 \), as long as \( \lambda \) is not an eigenvalue of (1.1). For simplicity we will assume that \( y(t) \) is odd and \( f(s) \) even. The above assumption presumes that the kernel \( K(s,t) \) is even in \( s \) and odd in \( t \). Under the above assumptions equation (2.1) is reduced to
(3.1) \[ \frac{2}{2n+1} \sum_{j=1}^{n} \frac{y_{n}(t_{j})t_{j}}{t_{j}^{2} - s_{k}^{2}} + \frac{2\lambda\pi}{2n+1} \sum_{j=1}^{n} K(s_{k}, t_{j})y_{n}(t_{j}) = f(s_{k}) \]

or using matrix notation

(3.2) \[ (A_{n} + \lambda C_{n})y = f \]

where \( A_{n}, y, f \) are defined in (2.5), and

(3.3) \[
C_{n} = \frac{2\pi}{2n+1} \begin{bmatrix}
K(s_{1}, t_{1}) & K(s_{1}, t_{2}) & \cdots & K(s_{1}, t_{n}) \\
K(s_{2}, t_{1}) & K(s_{2}, t_{2}) & \cdots & K(s_{2}, t_{n}) \\
& \ddots & \ddots & \ddots \\
K(s_{n}, t_{1}) & K(s_{n}, t_{2}) & \cdots & K(s_{n}, t_{n})
\end{bmatrix}
\]

From the general theory of integral equations ([6], p.338) we know that equation (3.1) is equivalent to a Fredholm Integral Equation

(3.4) \[ y(t) - \frac{\lambda}{\pi} \int_{1}^{t} (1-x^{2})^{-1/2} y(x) dx \int_{-1}^{1} \frac{K(s, x)(1-s^{2})^{1/2}}{s-t} ds = -f(t) + C \]

where

\[ F(t) = \frac{1}{\pi} \int_{-1}^{1} \frac{f(s)(1-s^{2})^{1/2}}{s-t} ds \]

and \( C \) is an arbitrary constant determined by the additional
condition (1.2). Since we assumed that \( y(t) \) is an odd function then \( C = 0 \).

We set

\[
L(x,t) = \int_1^1 \frac{k(s,x)(1-s^2)^{1/2}}{s-t} \, ds
\]

and approximate

\[
(3.6) \quad \int_1^1 (1-x^2)^{-1/2} y(x) \, L(x,t) \, dx = \frac{\pi}{2n+1} \sum_{j=1}^{2n+1} y_n(x_j) L(x_j,t),
\]

where

\[
x_j = \cos[(2j-1)\pi/(2n+1)], \quad j = 1, \ldots, 2n+1
\]

using Gauss-Chebyshev quadrature formula, and

\[
(3.7) \quad L(x_j,t_m) = \frac{\pi}{2n+1} \sum_{j=1}^{2n+1} \frac{k(s_k,x_j)(1-s_k^2)}{s_k - t_m}
\]

where

\[
s_k = \cos[(k\pi)/(2n+1)] \quad k = 1, \ldots, 2n
\]

\[
t_m = \cos[(2m-1)\pi/(2n+1)] \quad m = 1, \ldots, 2n+1
\]

using the Gauss formula (4.7) in [2].

Equation (3.4) then is approximated by

\[
(3.8) \quad y_n(t_m) = \frac{\lambda \pi}{(2n+1)^2} \sum_{j=1}^{2n+1} y_n(x_j) \sum_{k=1}^{2n} \frac{k(s_k,x_j)(1-s_k^2)}{s_k - t_m} = -F(t_m),
\]

\[
m = 1, \ldots, 2n+1.
\]
If we use the assumptions made at the beginning of this section, (3.8) becomes

\[
y_n(t_m) + \frac{4\lambda T}{(2n+1)^2} \sum_{j=1}^{n} y_n(x_j) \sum_{k=1}^{n} \frac{K(s_k, x_j) t_m (1-s_k^2)}{t_m^2 - s_k^2} = -F(t_m),
\]

\[m = 1, \ldots, n.\]

Since \(x_j = x_j\) we obtain from (3.9) the following algebraic system of equations,

\[
y_n(t_m) + \frac{4\lambda T}{(2n+1)^2} \sum_{j=1}^{n} y_n(x_j) \sum_{k=1}^{n} \frac{K(s_k, x_j) t_m (1-s_k^2)}{t_m^2 - s_k^2} = -F(t_m),
\]

\[m = 1, \ldots, n.\]

In a matrix notation (3.10) will be

\[
(I + \lambda Q_n) y = F
\]

where

\[
Q_n = \frac{4\lambda T}{(2n+1)^2}
\]

\[
\begin{bmatrix}
q_1,1 & q_1,2 & \cdots & q_1,r \\
q_2,1 & q_2,2 & \cdots & q_2,r \\
\vdots & \vdots & \ddots & \vdots \\
q_n,1 & q_n,2 & \cdots & q_n,n
\end{bmatrix}
\]
and
\[
q_{i,j} = \sum_{k=1}^{n} \frac{K(s_k, \sigma_j)}{2 - s_k^2} (1 - s_k^2)
\]
\[\text{for } i = 1, \ldots, n, \quad j = 1, \ldots, n,
\]
\[
y = [y(t_1), \ldots, y(t_n)], \quad F = [F(t_1), \ldots, F(t_n)].
\]

We know from the standard Fredholm theory that, except for those \( \lambda \) which are eigenvalues of (3.3), the matrix \((I + \lambda C_n)\) has a bounded inverse for sufficiently large \( n \) (see [3] pp 335, [7]).

We also observe that the matrix
\[
Q_n = A_n^{-1} C_n
\]
which in turn implies that \((I + \lambda A_n^{-1} C_n)\) is nonsingular.

Since \( A_n \) is nonsingular, then the product
\[
\lambda_n (I + \lambda A_n^{-1} C_n) = A_n + \lambda C_n
\]
is nonsingular, for sufficiently large \( n \).

The last relation (3.14), coupled with the fact that equations
(1.1) and (3.3) have the same eigenvalues, shows that the Gauss-
Chebyshev method (3.1) possesses a unique solution as long as \( \lambda \) is not
an eigenvalue of (1.1). Moreover, it is easily seen that (3.11) and
(3.2) are equivalent in a sense that (3.11) is obtained from (3.2) by
multiplying it with \( A_n^{-1} \), and (3.2) is obtained from (3.11) by
multiplying it with \( A_n \).

In addition we also see that for the solution of equation
(3.2) we require much less computations than (3.11). This explains in part the popularity of the direct numerical methods (3.2) versus nondirect methods (3.10) or a similar one given in [7].

Conclusion

We have shown that the coefficient matrix obtained in the Gauss-Chemyshnev method is nonsingular in the case that \( y(t) \) is odd and \( f(t) \) even. A similar result can be obtained by following the same analysis, in the case that \( y(t) \) is even and \( f(t) \) is odd.

We have not investigated the case of an arbitrary \( y(t), f(t) \) and \( \lambda \neq 0 \), since in general symmetry in physical problems give rise mostly to equations that satisfy the assumptions made above.

A similar analysis may be possible for that case. Notice that the coefficient matrix of (2.1) with \( \lambda = 0 \), is nonsingular independently of \( y(t), f(t) \). Also observe that an expression of its inverse may be given by equation (3.3).

Acknowledgement

I would like to thank Prof. R. P. Srivastava for the communication of his results in [5].
References


