PRODUCT INTEGRATION METHODS FOR THE SOLUTION
OF SINGULAR INTEGRAL EQUATIONS OF CAUCHY TYPE

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ABSTRACT

A method for the numerical solution of singular integral equations
of Cauchy type is developed. The unknown function is expressed as a
product of a weight function and a continuous function \( \phi(t) \). The
continuous function \( \phi(t) \) is approximated by piecewise quadratic
polynomials, and the singular integral equation is reduced to a
linear algebraic system. Two numerical examples are given, and
comparisons are made with the widely used Gauss-type methods. By
comparing the stress intensity factors obtained in the solution of a
singular integral equation that arises in the analysis of a cruciform
crack, the superiority of the method developed here over Gauss-type
methods is demonstrated.
1. Introduction

A large class of mixed boundary value problems in Engineering can be reduced to a singular integral equation of the form:

\[(1.1) \quad \int_{-1}^{1} \frac{g(t)}{t-s} \, dt + \int_{-1}^{1} K(s,t)g(t) \, dt = f(s), \quad -1 < s < 1.\]

where \(g(t)\) is the unknown function, \(K(s,t)\) is the kernel which is bounded in \(-1 \leq s, t \leq 1\), and \(f(s)\) is a known input function. If we assume that \(K(s,t), f(s)\) are Hölder-continuous functions in \(-1 \leq s, t \leq 1\), then the index theorems of Noether [3] determines the admissible singular behavior, and, in principle, the solution can be obtained using properties of sectionally holomorphic functions [1]. In many cases a direct numerical solution of (1.1) may prove to be more efficient and practical. The index theory gives the weight function \(w(t)\), and \(g(t)\) in (1.1) is expressed by

\[(1.2) \quad g(t) = w(t)\phi(t)\]

where \(w(t)\) is generally of the form \((1+t)^{-\frac{1}{2}+\alpha}(1-t)^{-\frac{1}{2}+\beta}, \alpha, \beta = 0, \pm 1\), and \(\phi(t)\) is a continuous function in \([-1,1]\). In this paper we will consider only the case of \(w(t) = (1-t^2)^{-\frac{1}{2}}\), since one can easily extend the method to cover the other cases (see [11]).

Erdogan [4] proposed a technique based on the approximation of \(\phi(t)\) in (1.2) by Jacobi polynomials orthogonal to the weight function \(w(t)\). For \(w(t) = (1-t^2)^{-\frac{1}{2}}\), \(\phi(t)\) is approximated by \(\phi(t) = \sum_{n=0}^{N} a_n T_n(t)\), where \(T_n(t)\) is
the Chebyshev polynomial of degree $n$, and substituting in (1.1) a system of algebraic equations is obtained. In [4], [5] it is shown that the singular integral in (1.1) is exact for all $\phi(t)$, if $\phi(t)$ is a polynomial of degree $\leq 2N$, and if the node points are chosen so that $T_N(t_k) = 0$, and the collocation points satisfy $u_{N-1}(s_\alpha) = 0$, $u_{N-1}$ is the Chebyshev polynomial of the second kind. Since the values of $\phi(t)$ at different points than the ones given by $T_N(t_k) = 0$ may be needed, say $\pm 1$ ($\phi(\pm 1)$ represents the stress intensity factor in Fracture mechanics), an extrapolation is required. This introduces a new error which counterbalances the accuracy of the quadrature formula.

Theocaris and Ioakimides [6] used a Lobatto-Chebyshev type formula to avoid the extrapolation at $\pm 1$. Since again the node points are determined from the equation $(1-t_k^2)u_{2N-2}(t_k) = 0$, one will have to use an interpolation formula if the solution is needed for points different than $t_k$.

Product integration methods have been widely used in the solution of singular integral equations with logarithmic singular kernels (see Atkinson [8], [9]). An attempt was made in [10] to use product integration methods for the solution of equations of Cauchy type with closed contours.

Gerasoulis [11], and Gerasoulis and Srivastav [15], used a product integration method to solve (1.1) that was based on a piecewise linear approximation of the functions $\phi(t)$ and $K(s,t)\phi(t)$.

In this paper a product integration method based on quadratic piecewise polynomial approximation of $\phi(t)$ is presented. One of the advantages of this method over Gauss type methods is that the node points can be chosen arbitrarily, eliminating the need of an interpolation.
The approximation of the function \( \phi(t) \) with Chebyshev polynomials is appropriate in many circumstances, but the oscillatory nature of high-degree polynomials and the property that a fluctuation over a small portion of the interval \([-1,1]\) can induce large fluctuations over the entire range, restricts their use whenever we approximate functions that oscillate over a small part of \([-1,1]\).

It is well known that the use of piecewise polynomial approximation with mesh points concentrated over the small interval for which \( \phi(t) \) behaves badly, can avoid problems of the previous type. Since the method developed in the following sections is based on a piecewise polynomial approximation, we expect that it will be superior to the Gauss-type methods for cases in which the solution of the singular integral equation (1.1) changes erratically.

In section 2 the singular integral equation (1.1) is approximated by an algebraic system of linear equations, and in the following section 3, we show that the residual error of the approximation converges to zero, and also that the solution of the approximate linear system converges to the solution of the singular integral equation (1.1) under certain assumptions. Finally, in the last section two numerical examples are given. In the first example a singular integral equation with a known analytical solution is considered. The rate of convergence of the algorithm is observed to be \( O(h^3) \), which was expected since we have used a quadratic polynomial approximation. In the second example a singular integral equation that arises in the analysis of a cruciform crack is solved. By comparing the stress intensity factors obtained with the use of Gauss-Chebyshev and
Lobatto-Chebyshev methods (see [6]), and the piecewise quadratic polynomial approximation method, the superiority of the latter method is demonstrated.
2. Generalized Quadrature Formulae

For convenience, we rewrite (1.1) as

\[
\frac{1}{\pi} \int_{-1}^{1} \frac{K(s,t)g(t)}{t-s} \, dt = f(s), \quad -1 < s < 1,
\]

where \( K(s,t) \) is appropriately adjusted. Generally, the solution of (2.1) is not unique and an additional condition of the form

\[
\frac{1}{\pi} \int_{-1}^{1} g(t) \, dt = A, \quad \text{where } A \text{ is known}
\]

is needed.

Let us divide the interval \([-1,1]\) in 2n equal parts (the method is equally applicable with partitions involving unequal meshes). Define \( t_j = -1 + jh \), where \( j = 0, 1, \ldots, 2n \) and \( h = \frac{1}{n} \). Put \( g(t) = (1-t^2)^{-\frac{1}{2}} \phi(t) \) in (2.1) and use the Lagrange interpolation formula for three points to approximate,

\[
K(s,t)\phi(t) = \frac{(t-t_{2k})(t-t_{2k+1})}{(t_{2k+1}-t_{2k})(t_{2k+1}-t_{2k-1})} K(s,t_{2k}) \phi(t_{2k})
\]

\[
+ \frac{(t-t_{2k-1})(t-t_{2k+1})}{(t_{2k+1}-t_{2k})(t_{2k+1}-t_{2k-1})} K(s,t_{2k}) \phi(t_{2k-1})
\]

\[
(2.3)
\]

\[
+ \frac{(t-t_{2k-1})(t-t_{2k})}{(t_{2k+1}-t_{2k-1})(t_{2k+1}-t_{2k})} K(s,t_{2k}) \phi(t_{2k+1}), \quad \text{for}
\]

\( t_{2k-1} \leq t \leq t_{2k+1}, \quad k = 1, 2, \ldots, n. \)
After elementary integration, we find that (2.1) is approximated by

\[
\frac{1}{n} \sum_{j=1}^{n} \left[ a_j(s)K(s,t_{2j-2})\phi(t_{2j-2}) + b_j(s)K(s,t_{2j-1})\phi(t_{2j-1}) + c_j(s)K(s,t_{2j})\phi(t_{2j}) \right] = f(s),
\]

where the weights \( a_j(s), b_j(s), c_j(s) \) are given by:

\[
a_j(s) = \frac{1}{2h^2} \left[ (1-t_{2j}^2)^{\frac{1}{2}} + (1-t_{2j-2}^2)^{\frac{1}{2}} + [s^2 + t_{2j-1}^2 - s(t_{2j-1} + t_{2j-2})] \right],
\]

\[
b_j(s) = -\frac{1}{h^2} \left[ (1-t_{2j}^2)^{\frac{1}{2}} + (1-t_{2j-2}^2)^{\frac{1}{2}} + [s^2 + t_{2j-1}^2 - s(t_{2j-1} + t_{2j-2})] \right],
\]

\[
c_j(s) = \frac{1}{2h^2} \left[ (1-t_{2j}^2)^{\frac{1}{2}} + (1-t_{2j-2}^2)^{\frac{1}{2}} + [s^2 + t_{2j-1}^2 - s(t_{2j-1} + t_{2j-2})] \right],
\]

and

\[
A_j(s) = \frac{1}{\sqrt{1-s^2}} \ln \left| \frac{\Theta_{2j}^2 - \tan^2 \frac{\Theta_{2j}}{2}}{\Theta_{2j}^2 - \tan^2 \frac{\Theta_{2j}}{2}} \right|
\]

where

\[
\Theta_k = \arcsin t_k, \text{ for all } k.
\]
Using $g(t) = (1-t^2)^{-1/2}$ $\phi(t)$, and approximating $\phi(t)$ with a piecewise quadratic polynomial (i.e., setting $K(s,t)$ in (2.3)) equation (2.2) becomes

\begin{equation}
\frac{1}{n} \sum_{j=1}^{n} [a_j \phi(t_{2j-2}) + b_j \phi(t_{2j-1}) + c_j \phi(t_{2j})] = A
\end{equation}

where

\begin{align*}
a_j &= \frac{1}{2\pi^2} \left\{ \left( \frac{1}{2} t_{2j} t_{2j-1} \right) \theta_j + (t_{2j} + t_{2j-2}) \theta_j - \frac{1}{4}D_j \right\}, \\
b_j &= -\frac{1}{2\pi^2} \left\{ \left( \frac{1}{2} t_{2j} t_{2j-2} \right) \theta_j + (t_{2j} + t_{2j-2}) \theta_j - \frac{1}{4}D_j \right\}, \\
c_j &= \frac{1}{2\pi^2} \left\{ \left( \frac{1}{2} t_{2j-1} t_{2j-2} \right) \theta_j + (t_{2j-1} + t_{2j-2}) \theta_j - \frac{1}{4}D_j \right\},
\end{align*}

(2.7)

\begin{align*}
\theta_j &= \theta_{2j} - \theta_{2j-2}, \\
\theta_j &= \cos \theta_{2j} - \cos \theta_{2j-2}, \\
D_j &= \sin 2\theta_{2j} - \sin 2\theta_{2j-2},
\end{align*}

where once again
\[ G_k = \arcsin k \text{ for all } k. \]

For convenience, we rewrite (2.4) and (2.6) as

\begin{equation}
\frac{1}{2\pi^2} \sum_{j=0}^{n} w_i(s)K(s,t_j)\phi(t_j) = f(s)
\end{equation}

\begin{equation}
\frac{1}{2\pi^2} \sum_{j=0}^{n} v_j \phi(t_j) = A
\end{equation}
where $w_0(s) = a_1(s)$

$$w_i(s) = \begin{cases} \frac{c_i(s) + a_{i+2}(s)}{2} & \text{if } i \text{ is even} \\ \frac{b_{i+1}(s)}{2} & \text{if } i \text{ is odd} \end{cases}$$

(2.9)

$$w_{2n} = c_n(s)$$

and $v_0 = a_1$

$$v_i = \begin{cases} \frac{c_i + a_{i+2}}{2} & \text{if } i \text{ is even} \\ \frac{b_{i+1}}{2} & \text{if } i \text{ is odd} \end{cases}$$

(2.10)

$$v_{2n} = c_n$$

Let us choose $2n$ collocation points $s_k$ such that $t_k \leq s_k \leq t_{k+1}$, $k = 0, 1, \ldots, 2n-1$. Then the functional equations (2.8) are reduced to a linear algebraic system of equations with $(2n+1)$ unknowns $\phi(t_i)$ and $2n+1$ equations,

$$\frac{1}{\pi_i} \sum_{i=0}^{2n} w_i(s_k) K(s_k, t_i) \phi(t_i) = f(s_k), k = 0, 1, 2, \ldots, 2n-1$$

(2.11)

$$\frac{1}{\pi_i} \sum_{i=0}^{2n} v_i \phi(t_i) = A$$
If instead of rewriting (1.1) as (1.2) and after approximating (1.2), we were approximating (1.1) directly using piecewise quadratic polynomials for \( \phi(t) \) and \( K(s,t)\phi(t) \), we would have obtained the following functional equation,

\[
\frac{1}{\pi} \sum_{i=0}^{2n} [w_i(s) \pi v_i K(s,t_i)] \phi(t_i) = f(s)
\]

(2.11a)

\[
\sum_{i=0}^{2n} v_i \phi(t_i) = A.
\]

(2.11b)

After choosing the collocation points \( s_k, k = 0, \ldots, 2n-1 \), (2.11a) becomes

\[
\frac{1}{\pi} \sum_{i=0}^{2n} [w_i(s_k) \pi v_i K(s_k,t_i)] \phi(t_i) = f(s_k)
\]

(2.12)

\[
\sum_{i=0}^{2n} v_i \phi(t_i) = A,
\]

where \( w_i(s_k), v_i \) are defined in (2.9) and (2.10).

The singular integral equations (2.1) and (1.1) are in general equivalent, but the approximating algebraic systems (2.11) and (2.12) are not equivalent. The solution of either (2.11) or (2.12) gives us a numerical estimation of the solution of the singular integral equation (1.1).
3. Error Analysis and Convergence of the Algorithm

In this section we will show that the approximate functional equation (2.11a) is consistent with the singular integral equation (1.1) (i.e. the truncation error converges to zero as the step size \( h \) becomes small). Also, we will show that under certain assumptions the numerical solution converges to the solution of the singular integral equation as \( n \) becomes large. To do this we will introduce the following linear operators:

\[
(M\phi)(t) = \frac{1}{\pi} \int_{-1}^{1} \frac{(1-s^2)^{\frac{1}{2}} \phi(s)ds}{s-t} ,
\]

\[
(T\phi)(s) = \frac{1}{\pi} \int_{-1}^{1} \frac{\phi(t)dt}{(1-t^2)^{\frac{1}{2}}(t-s)} ,
\]

\[
(K\phi)(s) = \int_{-1}^{1} (1-t^2)^{-\frac{1}{2}} K(s,t)\phi(t)dt ,
\]

where \( \phi \in H_\alpha \), \( H_\alpha \) is the Hölder space with index \( \alpha \) (for more details, see Muskhelishvili [17]). \( MT \) maps \( H_\alpha \) into itself, while \( K \) maps the space of continuous functions \( C[-1,1] \) into \( H_\alpha \). If \( \phi \in C[-1,1] \), then the infinity norm is defined by

\[
\| \phi \|_\infty = \sup_{t \in [-1,1]} |\phi(t)| ,
\]

while if \( \phi \in H_\alpha \), we introduce the Holder norm by.

\[
\| \phi \|_{H_\alpha} = \| \phi \|_\infty + \sup_{t+s \in [-1,1]} \frac{|\phi(t)-\phi(s)|}{|t-s|^{\alpha}} .
\]
In [7] it is shown that $H_\alpha$ is a Banach space with a norm defined in (3.3). The norms of the operators in (3.1) are defined by

$$\|M\| = \sup_{\phi \in H_\alpha, \phi \neq 0} \frac{\|M\phi\|_\infty}{\|\phi\|_{H_\alpha}}$$

(3.4) $$\|T\| = \sup_{\phi \in H_\alpha, \phi \neq 0} \frac{\|T\phi\|_\infty}{\|\phi\|_{H_\alpha}}$$

$$\|K\| = \sup_{\phi \in \mathcal{C}[-1,1], \phi \neq 0} \frac{\|K\phi\|_{H_\alpha}}{\|\phi\|_\infty}$$

(See also [7] for similar definition.)

We can easily show that

$$\|M\| \leq C_1 + C_2,$$

where

(3.5) $$C_1 = \max_{t \in [-1,1]} \frac{1}{\pi} \int_{-1}^{1} |t-s|^{-1}(1-s^2)^{\frac{1}{2}} ds$$

$$C_2 = \max_{t \in [-1,1]} \frac{1}{\pi} \int_{-1}^{1} \frac{(1-s^2)^{\frac{1}{2}}}{s-t} ds,$$

and that $$\|K\| \leq 2C_3 \|K\|_{H_\alpha}, C_3 = \int_{-1}^{1} \frac{1}{(1-t^2)^{\frac{1}{2}}} dt = \pi.$$

Using the definitions in (3.1), we can rewrite (1.1) as

(3.6) $$T\phi + K\phi = f(s)$$
Equation (3.6) is also equivalent to

(3.7) \[ \phi + MK\phi = Mf + C. \]

(See [1], [2], or [7]).

where C is a constant determined by (2.2), i.e.

\[
\frac{1}{\pi} \int_{-1}^{1} \frac{\phi(t)}{(1-t^2)^{\frac{3}{2}}} \, dt = A.
\]

Let us consider the sequence of functions \( \{\phi_n(t)\}_{n=1}^{\infty}, \{[K(s,t)\phi(t)]_n\}_{n=1}^{\infty} \) that approximate \( \phi(t), K(s,t)\phi(t) \) respectively, and define

(3.8) \[ \epsilon_n(t) = |\phi(t) - \phi_n(t)|, \epsilon_n(t,s) = |K(s,t)\phi(t) - [K(s,t)\phi(t)]_n|. \]

If \( \phi_n(t), [K(s,t)\phi(t)]_n \) are the piecewise quadratic polynomials approximating the continuous functions \( \phi(t), K(s,t)\phi(t) \) respectively, then \( \epsilon_n(t), \epsilon_n(t,s) \) converge to zero uniformly. Uniform convergence of a sequence of continuous functions is, in general, not enough to yield a convergent sequence of the Cauchy principal value integral \( T\phi_n \) (see [14]).

Stewart [14] has shown that for continuously differentiable functions, there exists a uniformly convergent approximating sequence of piecewise linear functions for which the Cauchy principal value integral \( T\phi_n \) converges uniformly. Here we will make the assumption that \( \phi \in C^3[-1,1] \) (three times continuously differentiable). Since if \( \phi \in C^2[-1,1] \) but not in \( C^3[-1,1] \), then one should use a piecewise linear approximation (see [11], [15]), instead of a piecewise quadratic approximation, because the additional computational effort will not guarantee any better convergence. We are now ready to present the following Theorem.
Theorem 1. If $\phi \in C^3([-1,1])$ and $\phi_n(t)$ is given in (2.3) (i.e. set $K(s,t)=1$), then

\begin{align*}
(i) \quad & \|e_n\|_\infty \leq \frac{h^3}{6} \|\phi'''\|_\infty \\
(ii) \quad & \sup_{t+s} \left| \frac{e_n(t) - e_n(s)}{t-s} \right| \leq \beta \ h^3 \|\phi'''\|_\infty \\
& \text{for } t,s \in [-1,1]
\end{align*}

where $\beta$ is a constant independent of $h$.

(iii) $T e_n$ converges to zero as $n$ becomes large.

Proof. (i), (ii) are obvious (see [10]).

(iii) We may write

\[
T e_n = \frac{1}{\pi} \int_{-1}^{1} \frac{(1-t^2)^{1/2} e_n(t)}{t-s} \, dt = \\
\frac{1}{\pi} \int_{-1}^{1} (1-t^2)^{-1/2} \frac{e_n(t) - e_n(t)}{t-s} \, dt
\]

and use (ii) with $\alpha=1$ to conclude that $T e_n$ converges to zero as $n \to \infty$.

Finally, since the truncation error of the approximation of (3.6) is given by

\[
T e_n + K e_n, \quad \text{where}
\]

(3.9) \quad $K e_n = \int_{-1}^{1} (1-t^2)^{-1/2} K(s,t) e_n(t) \, dt$

we can easily see that the approximate equation of (3.6) is consistent with (3.6).
In the rest of this chapter we will show that $\phi_n + \phi$ under certain assumptions. Let us assume that the solution of the linear system (2.9) is $\phi^*(t_0), \phi^*(t_1), \ldots, \phi^*(t_n)$ for some given $n$. Then define $\phi_n^*(t)$ and 
$[K(s,t)\phi_n^*(t)]_n$, using $\phi^*(t_i)$ and the piecewise quadratic approximation formulas (2.3).

Let us define a function $f_n(s)$ by

(3.10) $T\phi_n^* + K\phi_n^* = f_n(s)$

From the definition of $f_n(s)$, we can see that $f_n(s_k) = f(s_k)$ ($s_k$ are the collocation points). Since

$\phi_n^* \in H_a$, $K\phi_n^* \in H_a$ and $T\phi_n^* \in H_a$, then $f_n(s) \in H_a$.

Equation (3.10) is rewritten as

(3.11) $\phi_n^* + MK\phi_n^* = Mf_n + C_1$

and since we have assumed that

$$\frac{1}{\pi} \int_{-1}^{1} (1-t^2)^{-\frac{1}{2}} \phi_n^*(t) dt = A,$$

we can easily show that $C_1$ is equal to the constant $C$ of equation (3.7).

If we subtract (3.11) from (3.7), we obtain

$$\phi - \phi_n^* + M(K\phi - K_n\phi + K_n\phi - K_n\phi_n^*) = M(f - f_n)$$

so that

(3.12) $\phi - \phi_n^* + MK_n(\phi - \phi_n^*) = M(f - f_n) - M(K - K_n)\phi$.
We have assumed that $I + MK$ possesses an inverse. Using the theory of collectively compact operators and the result on page 97 of [9], it follows that if $||K-K_n|| \to 0$ as $n \to \infty$, then $I+MK_n$ possesses an inverse as $n \to \infty$. If we let $B_n = (I+MK_n)^{-1}$, then $B_n$ maps $C[-1,1]$ into $C[-1,1]$, and we can define the norm for $B_n$, (see [7])

$$||B_n|| = \sup_{g \in C[-1,1]} ||B_n g||/||g||_{\infty}$$

We rewrite (3.12) as

$$\phi - \phi_n^* = B_n (f-f_n) - B_n (K-K_n) \phi$$

and after taking the infinite norm in both sides, we obtain

$$||\phi - \phi_n^*||_{\infty} \leq ||B_n|| \ ||M|| \ ||f-f_n||_{H_{\alpha}} + ||B_n|| \ ||M|| \ ||K-K_n|| \ ||\phi||_{\infty}$$

Now we are ready to present the main theorem of this section.

Theorem 2 Suppose that $f, K, K_n$ are in $H_{\alpha}$ and that (1.1) possesses a solution. If

$$\lim_{n \to \infty} ||f-f_n||_{H_{\alpha}} = 0, \lim_{n \to \infty} ||K-K_n|| = 0$$

then

$$\lim_{n \to \infty} ||\phi - \phi_n^*||_{\infty} = 0.$$

Proof It is easily seen from (3.13).
4. Numerical Computations and Examples

1) We first consider the singular integral equation

\[ \int_{-1}^{1} \frac{g(t)}{t-s} \, dt = U_{m-1}(s), \quad m = 1, 2, \ldots \]  

(4.1)

The analytical solution of equation (4.1) is \( g(t) = (1-t^2)^{-k} \), if the following additional condition is imposed

\[ \int_{-1}^{1} g(t) \, dt = 0. \]

\( T_m(t), U_m(t) \) are the Chebyshev polynomials of degree \( m \) of the first and second kind respectively. To solve (4.1) numerically, we put

\( g(t) = (1-t^2)^{-k} \rho(t) \) and use the methods developed in section 2. We set \( K(s_k, t_1) = 0 \) and \( f(s_k) = U_{m-1}(s_k) \) in (2.12) or \( K(s_k, t_1) = 1 \) and \( f(s_k) = U_{m-1}(s_k) \) in (2.11) and solve the resulting algebraic linear system. For \( m = 3 \) and \( s_k = t_k + \frac{n}{2}, \quad k = 0, 1, \ldots, 2n-1 \) the numerical solution is shown in Table 1. Since \( \rho(t) \) is odd, we only show the solution for selected positive node points \( t_i, \quad i=0, \ldots, n \), and \( n \) is the number of node points in \([0, 1]\).

**TABLE 1**

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</tr>
</tbody>
</table>
The effect of the collocation points $s_k$ on the convergence of the numerical method is insignificant. It has been observed that the accuracy of $\hat{\phi}(t)$ changes only slightly, for different choices of collocation points $s_k$, such that $t_k \leq s_k \leq t_{k+1}$.

The order of convergence is generally defined by $O(h^p)$, where

$$
\frac{e_{2n}}{e_n} = -p\ln 2 \quad \text{and} \quad e_n = |\hat{\phi}(t) - \phi_n(t_k)|.
$$

It is easily seen from Table 1 that $p = 3$ and the order of convergence is $O(h^3)$. This is an expected result since the algorithm is based on a piecewise quadratic approximation of the function $\phi(t)$.

For a comparison, we give in Table 2 the numerical solution of (4.1) at $t=1.00$ for the piecewise linear approximation of $\phi(t)$ (see [15]). Once again, we notice that the order of convergence of $\phi_n$ is the expected one, i.e. $O(h^2)$.

**TABLE 2**

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\phi_n(1.00)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.91477</td>
</tr>
<tr>
<td>9</td>
<td>0.97748</td>
</tr>
<tr>
<td>17</td>
<td>0.99419</td>
</tr>
<tr>
<td>33</td>
<td>0.99852</td>
</tr>
</tbody>
</table>

ii) As a second example, we consider the singular integral equation

$$
(4.2) \quad \frac{1}{\pi} \int_{-1}^{1} \left[ \frac{1}{t-s} + \frac{t(t^2-s^2)}{(t^2+s^2)^2} \right] \phi(t) \, dt = 1
$$

which arises in the analysis of a cruciform crack (see [6]). The unknown
function $g(t)$ is proportional to the dislocation density along the crack branches. The solution is an odd function. Equation (4.2) has been solved numerically in [6] using the Gauss-Chebyshev and Lobatto-Chebyshev methods. We set

$$K(s,t) = \frac{t(t^2-s^2)}{(t^2+s^2)^2}, \quad f(s) = 1 \text{ in (2.12) and choose}$$

$$s_k = t_k + \frac{h}{2}, \quad k = 0, \ldots, n.$$  

Table 3 shows the numerical solution of (4.1) for positive node points $t_k$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$n$</th>
<th>5</th>
<th>9</th>
<th>17</th>
<th>33</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>-0.87 x $10^{-16}$</td>
<td>-0.62 x $10^{-16}$</td>
<td>-0.57 x $10^{-14}$</td>
<td></td>
</tr>
<tr>
<td>0.2500</td>
<td>-0.01105</td>
<td>-0.01663</td>
<td>-0.01409</td>
<td>-0.01394</td>
<td></td>
</tr>
<tr>
<td>0.5000</td>
<td>0.19468</td>
<td>0.19513</td>
<td>0.19531</td>
<td>0.19521</td>
<td></td>
</tr>
<tr>
<td>0.7500</td>
<td>0.52348</td>
<td>0.52349</td>
<td>0.52332</td>
<td>0.52332</td>
<td></td>
</tr>
<tr>
<td>1.0000</td>
<td>0.86570</td>
<td>0.86367</td>
<td>0.86354</td>
<td>0.86354</td>
<td></td>
</tr>
</tbody>
</table>

It is mentioned in section 2 that the proposed numerical methods (2.11) and (2.12) are not equivalent. To see that, we have solved equation (2.11) with

$$K(s,t) = 1 + \frac{t(t+s)(t-s)^2}{(t^2+s^2)^2}, \quad f(s) = 1,$$

and the numerical solution at $t = 1.00$ is shown in Table 4. Also, to see the effect of the choice of the collocation points on the numerical solution of (4.2), we give in Table 4 the results for two sets of
collocation points, namely $s_k = t_k + \frac{h}{2}$ and $s_k = t_k + 5 \times 10^{-6}h$, $k = 0, 1, \ldots, n$.

TABLE 4

<table>
<thead>
<tr>
<th>$\phi_n(1.00)$ (eq. 2.11)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
</tr>
<tr>
<td>-----</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>9</td>
</tr>
<tr>
<td>17</td>
</tr>
<tr>
<td>33</td>
</tr>
</tbody>
</table>

From Table 4, we can see now that the effect of the collocation points on the numerical solution is not significant. Also, we can see from Tables 3 and 4 that (2.12) gives better results than (2.11). This is because we have approximated two different kernels $K(s,t)$ in (2.11) and (2.12) and the accuracy for each approximation is different.

The value $\phi(1)$ represents the stress intensity factor and it is interesting to see $\phi(1)$ obtained by different methods (see Table 5). The numerical results for Gauss-Chebyshev and Lobatto-Chebyshev have been reproduced from [6].
### TABLE 5

<table>
<thead>
<tr>
<th>Method</th>
<th>Gauss-Chebyshev</th>
<th>Lobatto-Chebyshev</th>
<th>Piecewise Quadratic (eq. 2.12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>( \phi(t) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.83635</td>
<td>0.85970</td>
<td>0.90034</td>
</tr>
<tr>
<td>4</td>
<td>0.83882</td>
<td>0.86387</td>
<td>0.86970</td>
</tr>
<tr>
<td>5</td>
<td>0.86289</td>
<td>0.86449</td>
<td>0.86570</td>
</tr>
<tr>
<td>6</td>
<td>0.86381</td>
<td>0.86441</td>
<td>0.86455</td>
</tr>
<tr>
<td>7</td>
<td>0.86528</td>
<td>0.86424</td>
<td>0.86397</td>
</tr>
<tr>
<td>8</td>
<td>0.86282</td>
<td>0.86408</td>
<td>0.86380</td>
</tr>
<tr>
<td>9</td>
<td>0.86603</td>
<td>0.86396</td>
<td>0.86367</td>
</tr>
<tr>
<td>10</td>
<td>0.86283</td>
<td>0.86387</td>
<td>0.86363</td>
</tr>
<tr>
<td>11</td>
<td>0.86464</td>
<td>0.86380</td>
<td>0.86359</td>
</tr>
</tbody>
</table>

It is obvious from Table 5 that the piecewise quadratic method converges much faster and more smoothly than the Gauss-Chebyshev and Lobatto-Chebyshev methods. The correct value up to four significant figures is given in Rooke and Sneddon [13], and is \( \phi(1) = 0.8636 \). It should be mentioned here that in Gauss-Chebyshev and Lobatto-Chebyshev methods, there are \( 2n \) node points in \([-1,1]\) while in the piecewise quadratic method, there are \( 2n-1 \) node points. So as a conclusion, the piecewise quadratic method converges even faster than the Table 5 shows.

All the calculations have been performed in IBM-370 with a double precision arithmetic.
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