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APPLICABILITY OF INCREMENTAL ITERATIVE ALGORITHMS

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TABLE OF CONTENTS

KEY WORDS: 1
ABSTRACT 2
  1. INTRODUCTION 3
    2. Basic Definitions And Properties—Limits 6
      2.0.1. The Limit Is the Minimum Stable State Of A Network 9
      2.1. Incremental Changes In A Network 11
References 17
Index 19
LIST OF FIGURES

Figure 1: Weighted Digraph And Equations Giving Cost Of Minimum Path 3
ABSTRACT

In computer science there has been much interest in iteration as a procedure for obtaining the solution of a system of equations. The applicability of iteration does not depend strongly on the form or properties of the system of equations. Its use ranges from solution of numerical equations to data-flow analysis. Also the problem of finding incremental algorithms which adjust a solution to a small change in parameters has received much attention recently and is of particular importance for large problems such as arise in data-flow analysis.

In this paper we show that one cannot always continue iterating from a previous solution after even a small change in parameters; we give conditions under which it is legitimate to do so. This result follows a brief discussion of iteration in general.

We consider finding a fixed point, $X$, by iteration of a monotonic function, $W$, on a partially ordered set $Q$. This differs somewhat from previous developments in that, beyond monotonicity, $W$ is no further constrained, $Q$ need not be a semi-lattice, and $X$ need not be reachable by a finite number of iterations.
1. INTRODUCTION

Consider a collection of functions \( W_p(X) = W(X, P) \), parametrized by the set of parameters \( P \), where \( W \) has properties making an iterative solution for a distinguished fixed point \( X_p \) feasible. Suppose also that one has determined the distinguished fixed point \( X_A \) of the equation \( X = W_A(X) \) and that we wish to solve the related equation \( X = W_b(X) \) for a different set of parameters \( B \). In this paper we give conditions on the relation of \( A \) and \( B \) which make it possible to solve the related equation \( X = W_b(X) \) beginning at \( X_A \).

This result on incremental iterative algorithms follows a brief discussion of iteration in general. We consider finding a fixed point, \( X \), by iteration of a monotonic function, \( W \), on a partially ordered set \( Q \). This differs somewhat from previous developments [Birkhoff 67, Hecht 77] in that, beyond monotonicity, \( W \) is no further constrained, \( Q \) need not be a semi-lattice, and \( X \) need not be reachable by a finite number of iterations.

In computer science there has been much interest in iteration as a procedure for obtaining the solution of a system of equations, especially in the area of program analysis [Fong 75, Kildall 73, Kam 76]. Information about the definition and use of data in a program can be expressed as the solution of a set of equations [Cocke 70]. Iteration is widely used since its applicability does not depend strongly on the form or properties of the system of equations.

First we give an example of a function, \( W \), whose value can be determined by iteration, then we develop general definition, conditions, and properties of such functions, finally the incremental properties are studied.

\[ \downarrow \text{ means minimum} \]

\[
\begin{align*}
&\begin{array}{c}
v1 & \rightarrow & v2 & \rightarrow & v3 \\
& & | & & | \\
& & | & 1 & | \\
& & | & & | \\
& & | & / & / \\
& & | & & | \\
v6 & \rightarrow & v5 & \rightarrow & v3
\end{array}
\end{align*}
\]

\[
\begin{align*}
x_1 &= (1 + x_2) \downarrow (4 + x_3) \\
x_2 &= (1 + x_4) \\
x_3 &= (1 + x_5) \\
x_4 &= (1 + x_3) \downarrow (4 + x_5) \\
x_5 &= (1 + x_2) \downarrow (4 + x_6) \\
x_6 &= 0
\end{align*}
\]

Figure 1: Weighted Digraph And Equations Giving Cost Of Minimum Path
To introduce basic notation and concepts, consider the problem of finding the minimum path from \( v_1 \) to \( v_6 \). We trace the solution of the minimum path equations (see [Aho Hopcroft Ullman 75], section 5.6) in figure 1 by iteration and then increment or change parameters and retrace the solution.

Define:

\[
X = \langle x_1, \ldots, x_6 \rangle
\]

\[
X^0 = \langle 0, \ldots, 0 \rangle
\]

To get \( W^0 \) substitute \( \langle 0, 0, 0, 0, 0, 0 \rangle \) for \( \langle x_1, \ldots, x_6 \rangle \) and solve:

\[
\begin{align*}
x_1 &= (1 + 0) + 4 + 0 = 1 \\
x_2 &= (1 + 0) = 1 \\
x_3 &= (1 + 0) = 1 \\
x_4 &= (1 + 0) + (4 + 0) = 1 \\
x_5 &= (1 + 0) + (4 + 0) = 1 \\
x_6 &= 0 = 0
\end{align*}
\]

We say:

\[
X^1 = W(X) = W(\langle 0, 0, 0, 0, 0, 0 \rangle) = \langle 1, 1, 1, 1, 1, 1 \rangle
\]

In a similar way:

\[
\begin{align*}
x_1^2 &= W^2(X) = W(\langle 1, 1, 1, 1, 1, 1 \rangle) = \langle 2, 2, 2, 2, 2, 2 \rangle \\
&
\end{align*}
\]

\[
\begin{align*}
x_4^i &= W^i(X) = W(\langle 3, 3, 3, 3, 3, 3 \rangle) = \langle 4, 4, 4, 4, 4, 4 \rangle
\end{align*}
\]

To get \( X^6 \) substitute \( \langle 4, 4, 4, 4, 4, 4 \rangle \) for \( \langle x_1, \ldots, x_6 \rangle \) and solve:

\[
\begin{align*}
x_1 &= (1 + 4) + 4 + 4 = 5 \\
x_2 &= (1 + 4) = 5 \\
x_3 &= (1 + 4) = 5 \\
x_4 &= (1 + 4) + (4 + 4) = 5 \\
x_5 &= (1 + 4) + (4 + 0) = 4 \\
x_6 &= 0 = 0
\end{align*}
\]

So:

\[
\begin{align*}
x_5^6 &= W^5(X) = W(\langle 4, 4, 4, 4, 4, 4 \rangle) = \langle 5, 5, 5, 5, 5, 5 \rangle \\
x_6^6 &= W^6(X) = W(\langle 5, 5, 5, 5, 5, 5 \rangle) = \langle 6, 6, 6, 6, 6, 6 \rangle \\
x_7^6 &= W^7(X) = W(\langle 6, 6, 6, 6, 6, 6 \rangle) = \langle 7, 7, 7, 7, 7, 7 \rangle
\end{align*}
\]
Substituting \(<8, 7, 5, 6, 4, 0>\) for \(<x_1, \ldots, x_6>\) on the right we get:

\[
\begin{align*}
  x_1 &= (1 + 7) \div (4 + 5) &= 8 \\
  x_2 &= (1 + 6) &= 7 \\
  x_3 &= (1 + 4) &= 5 \\
  x_4 &= (1 + 5) \div (4 + 4) &= 6 \\
  x_5 &= (1 + 7) \div (4 + 0) &= 4 \\
  x_6 &= 0 &= 0
\end{align*}
\]

\[x^8 = W^8(x) = W(<8, 7, 5, 6, 4, 0>) = <8, 7, 5, 6, 4, 0>\]

and finally substituting \(<8, 7, 5, 6, 4, 0>\) for \(<x_1, \ldots, x_6>\) on the right we would get an identical set of equations and find that:

\[x^9 = W^9(x) = W(<8, 7, 5, 6, 4, 0>) = <8, 7, 5, 6, 4, 0> = x^8 = W^8(x)\]

We say \(<8, 7, 5, 6, 4, 0>\) is a fixed or stable point of \(W\).

Now suppose we wished to change the fourth equation to:

\[x_4 = (2 + x_3) \div (4 + x_5)\]

We could start all over again with a new iteration and substitute 0's for all \(x_i\)'s on the right and by successive iterations be guaranteed to arrive at the correct answer. Can we start with the solution we have already obtained and get the correct answer by iteration using the new equations—designated \(W^8\)?

Substituting \(<8, 7, 5, 6, 4, 0>\) for \(<x_1, \ldots, x_6>\) on the right we get:

\[
\begin{align*}
  x_1 &= (1 + 7) \div (4 + 5) &= 8 \\
  x_2 &= (1 + 6) &= 7 \\
  x_3 &= (1 + 4) &= 5 \\
  x_4 &= (2 + 5) \div (4 + 4) &= 7 \\
  x_5 &= (1 + 7) \div (4 + 0) &= 4 \\
  x_6 &= 0 &= 0
\end{align*}
\]

So:

\[W^8(<8, 7, 5, 6, 4, 0>) = <8, 8, 5, 7, 4, 0>, \text{ and finally:}\]

\[W^8(<8, 8, 5, 6, 4, 0>) = <8, 8, 5, 7, 4, 0>\]

We say \(<8, 8, 5, 7, 4, 0>\) is a fixed or stable point of \(W^8\).

This is the same solution we would get, with many more iterations, if we had started with all 0's. (In fact with this example we would have gotten the same answer, no matter where we had started. Unfortunately this is not always so.)
2. Basic Definitions And Properties--Limits

The set Q which we now define will form the domain and range of the function for whose fixed points we will solve.

Q is a bounded set if:
1. Q is partially ordered by the relation ≤.

2. Q has a least element designated 0, i.e., ∀ X ∈ Q: 0 ≤ X.

Usually Q is a set of n-vectors and W is a system of n equations in n unknowns as in our example. Further, the relation as well as the zero of Q arise from those of the components and the components and coefficients are of the same type, e.g., all real numbers or all sets. But the results do not require any of these restrictions. In the context of a system of equations, we define the dependency graph of a system of equations to be a digraph with vertices = the set of variables, and an edge (i, j) if X_i occurs in the right-hand side of the equation for X_j (where i can equal j).

Next we consider useful constraints on the function W, which, as we said, can be thought of as the effect on the vector of values of x_i of plugging in values for x_i on the right of a set of equations of the form:

\{x_i = w(x_i, x_i_1, x_i_2, \ldots, x_i_n) \mid i \in \mathbb{N}^n\} \quad \text{where } x_i \in \{x_i \mid i \in \mathbb{N}^n\}

In effect all the results we develop for a W of this kind imply that all x_i's with 1 ≤ i ≤ n are evaluated simultaneously—the next set of x_i's are not plugged into the right side until all have been computed from the current set. But in fact, it is not difficult to show that the results continue to hold even if we do not require such simultaneity—which is the way such iterations are usually implemented, provided only weak conditions hold for W and Q.

A function W is called a bounded function if:

1. W:Q → Q and Q is a bounded set and

2. W is monotonic, i.e., if A and B are in Q and A ≤ B then W(A) ≤ W(B)

We represent n applications of W as follows:
\[ W_0^0(X) = X, \]
\[ W_n^0(X) = W(W_{n-1}^0(X)). \]

(W \( ^n \) (X) is also written W(X))

Notice that monotonicity does not mean that W(X) is necessarily \( \geq X \). On the other hand it does imply that:

**Lemma 2.1:** If \( W \) is monotonic then \( \forall j \in \mathbb{N}_0: W^{i+1}(0) \geq W^i(0) \).

**Proof:** \( W^1(0) \geq 0 \) so \( W^2(0) \geq W^1(0) \) and so \( \forall j \in \mathbb{N}_0: W^{i+1}(0) \geq W^i(0) \).

In fact if, for any \( X \), \( W(X) \) is related (is \( \geq \) or \( \leq \)) to \( X \), \( W^{i+1}(X) \) will be similarly related to \( W^i(X) \) and to \( X \).

Unless otherwise specified we assume that the function \( W \) referred to in this section is bounded.

\( W \) is stable at \( X \), or \( X \) is a stable state (of \( W \)) or \( X \) is a fixed point of \( W \) iff \( X = W(X) \)

**Lemma 2.2:** If \( W \) is a bounded function and \( W \) is stable at \( X \) then \( \forall j \in \mathbb{N}_0: W^j(0) \leq X \).

**Proof:** For any \( X, \ 0 \leq X \). So, by monotonicity, \( \forall j \in \mathbb{N}_0: W^j(0) \leq W^j(X) = X \).

A network, NET consists of a pair \((Q,W)\), where \( Q \) is a bounded set and \( W \) a bounded function. (We will sometimes speak of the "network \( W \)", meaning the network whose bounded function is \( W \).)

Given a network, \( NET = (Q,W) \), if \( X \in Q \):
- \( X \) is stable (in \( NET \)) if it is stable in \( W \).
- \( X \) is infinite (in \( NET \)) if \( \forall j \in \mathbb{N}_0: W^j(0) < X \)
- \( X \) is finite (in \( NET \)) if \( \exists j \in \mathbb{N}_0: W^j(0) \geq X \)

(Consider the monotonic function \( f(j) = W^j(0) \) from \( \mathbb{N}_0 \) into \( Q \). Intuitively, \( X \) is finite if \( X \) is in some interval \([0,j]\) = \{\( x \in Q \mid 0 \leq x \leq W^j(0) \}\}; \( X \) is infinite if \( X \) is unreachable by \( W \), i.e., no such interval contains \( X \) and \( X \) is larger than each such point. Notice that there can exist members of \( Q \) that are neither finite nor infinite.)
We say that \( \lim \) (finite) \( W'(0) = L \) or that \( W \) seeks (from 0) \( L \) if \( W \) is stable at \( L \) and either:

1. Finite Limit: \( L = W'(0) \) for some finite \( j \) in \( N \)

2. Infinite Limit: (Effectively we want \( L = W^\infty(0) \); we take \( L = \text{lub}\{W'(0)\} \).)
   a. \( L \) is infinite
   b. For \( X \in Q \): If \( X \) is infinite then \( X \geq L \). (This condition assures the uniqueness of the infinite limit)

We present some examples of finite and infinite limits:

Ex. 2.1: For \( Q = \text{Sets}, \leq \text{given by} \subseteq, \text{and} \ 0 = \emptyset \), let \( a \) and \( b \) be two fixed sets and
\[
W(X) = X \cap a \cup b.
\]
Then \( W'(0) = \emptyset; W^2(0) = W'(0) = b \), so \( L = b \) is the finite limit of this network. (Note that any subset of \( b \) is finite, and any superset of \( b \) is infinite; any other set is neither finite nor infinite.)

Ex. 2.2: The equations in the example of section 1 provide an example of a system of equations in which the limit is finite but not given by the first application of \( W \).

Ex. 2.3: Let \( G \) be a directed graph with vertices \( V \) and edges \( E \). Let \( Q \) be the power set of \( V \times V \), ordered by inclusion, with \( 0 = \emptyset \). Let
\[
\text{AMB} = \{(x,z) \mid \exists y \ (x,y) \in A \land (y,z) \in B\},
\]
and
\[
W(X) = X \times E \cup E.
\]
It is easy to verify that \( L \) is the edge set of the transitive closure of \( G \), and that it requires \( k = \text{diameter of} \ G \) iterations to reach the stable state. Thus, if \( G \) is a finite graph, \( L \) is a finite limit.

Ex. 2.4: Let \( Q \) = sets of strings in a language, ordered by set inclusion, with \( 0 = \epsilon \), the empty string. Let \( \cup \) be the concatenation operator, and let \( A \subseteq Q \), and
\[
W(X) = A \cup X \cup A.
\]
Then, as is well known [Aho Ullman 72], \( L = A^\omega \). This is an infinite limit, since \( A \) is not reached by a finite number of applications of \( W \).

Ex. 2.5: Let \( f(x) \) be a real-valued function with some fixed point in an interval in which \( 0 < f'(x) \leq r < 1 \). Let \( Q = R \) with the usual order, and let \( 0 = x_0 \) = some initial guess to the left of the fixed point. Then the usual fixed point iteration \( W(x) = f(x) \) will in general have the fixed point as its infinite limit.
2.0.1. The Limit Is the Minimum Stable State Of A Network

We now develop a series of consequences of these definitions, which are interesting in themselves and will lead to the conclusion that the limit is the minimum stable state, and are needed to establish our results on incremental iterative algorithms.

Lemma 2.3: If \( L \) is a finite limit and \( L \geq X \) then \( \exists j \in N_0 \ni W^j(X) = L \).

Proof:
\[
L \geq X \geq 0
\]
So for all \( k \in N_0 \):
\[
W^k(L) \geq W^k(X) \geq W^k(0)
\]
But there is an \( z \) so that when \( k = z \), \( L = W^z(0) \)
\[
L = W^z(L) \geq W^z(X) \geq W^z(0) = L.
\]

Lemma 2.4: If \( L \) is a finite limit and \( X \) is finite then \( \exists j \in N_0 \ni W^j(X) = L \).

Proof: Follows directly from lemma 2.3, and definition of finite.

In some cases a network \( NET = (W,Q) \) has a finite stable state which is the only stable point in a finite domain \( Q \). In such a case starting at any \( X \) in \( Q \) a finite number of applications of \( W \) to \( X \) will produce that limit. The minimum path network will generally have this property, as will most networks which arise from a set of equations whose dependency graph has no cycles so there is reason to consider this property.

Lemma 2.5: Let \( NET = (Q,W) \) such that

1. \( Q \) is a finite set.
2. \( NET \) has a finite limit \( L = W^0(0) \), and
3. \( L \) is the only stable member of \( Q \).

Then if \( X \in Q \) and there is a \( Y \) in \( Q \) which is either \( \geq \) or \( \leq \) \( W(Y) \), and \( Y \geq X \) it follows that \( \exists j \in N_0 \ni W^j(X) = L \).

Proof:
\[
Y \geq X \geq 0
\]
So for all \( k \in N_0 \):
\[
W^k(y) \geq W^k(x) \geq W^k(0)
\]

But if \( k = z, L = W^z(0) \)
\[
W^z(y) \geq W^z(x) \geq W^z(0) = L.
\]

But \( Y \geq W(y) \geq W^2(y) \geq \ldots \geq W^q(y) \geq \ldots \) or
But \( Y \leq W(y) \leq W^2(y) \leq \ldots \leq W^q(y) \leq \ldots \)

But "\geq" in the first case and "\leq" in the second case
cannot hold for all \( q \in N_0 \), because \( Q \) is finite.

So there must be a \( u = W^u(y) = L \)
Thus for \( j > j + z \geq u \):
\[
W^{j+z}(y) = L \geq W^{j+z}(x) \geq L.
\]
So \( \exists j + z \in N_0 \Rightarrow W^{j+z}(x) = L \).

If the domain \( Q \) has an largest element, 1, such that \( 1 \geq \) every member of \( Q \), then
1 it will serve the in place of \( Y \) in the lemma above for any \( X \) in \( Q \). Alternatively \( X \) itself
would serve in place of \( Y \) if it were guaranteed either \( \geq \) or \( \leq W(X) \).

Even if a network domain is not finite, as long as sequences of increasing and
decreasing values are of finite length, the limit will be reached by application of \( W \) a finite
number of times to any member of \( Q \). This is stated more precisely with the following
definitions and lemma.

A partial ordering is said to satisfy the ascending chain condition if any
sequence of members \( a_1 < a_2 < \ldots < a_p \) is finite.

A partial ordering is said to satisfy the descending chain condition if any
sequence of members \( a_1 > a_2 > \ldots > a_p \) is finite.

Lemma 2.6: If \( NET = (Q,W) \) meets the same conditions as in lemma 2.5,
extcept that condition 1 is replaced with:

1. \( Q \) satisfies both the ascending and descending chain condition

Then if \( X \in Q \) and there is a \( Y \) in \( Q \) which is either \( \geq \) or \( \leq W(Y) \), and \( Y \geq X \)
it follows that \( \exists j \in N_0 \Rightarrow W^j(X) = L \).

Proof: Proof is identical to that of lemma 2.5, except that the finite chain
conditions rather than finiteness is used to argue that "\( \geq \)" nor "\( \leq \)" cannot hold
for all \( q \in N_0 \).
The next lemma will prove useful in determining the limit of a network.

**Lemma 2.7:** If a network has a finite stable state then it has a unique limit = its smallest stable state.

Proof: If \( X \) is any stable state then \( \forall j \in N_0: W^j(0) \leq X \), by 2.2. If \( X \) is finite stable state then, by definition, there is a \( j \) with \( X \geq W^j(0) \). Therefore for some \( j \), there is a finite stable state \( X = W^j(0) \). This is clearly a unique value.

**Lemma 2.8:** If a network has an infinite limit then that limit is unique and is its smallest stable state.

Proof: Follows from definition of infinite limit.

From the last two lemmas it follows immediately that:

**Corollary 2.0.1:** The limit of a network is its minimum stable state.

But notice that if a network has a number of stable states but none that are finite we cannot conclude that the smallest of these is the limit. This is because the prescribed ordering is only partial so there could be stable states which, though not finite, are also not infinite.

2.1. Incremental Changes in A Network

Often a network \( W \) function is the composition of some primitive functions operating on variables and some parameters. For example, in the minimum path example, figure 1, addition and minimum are the operators, and the coefficients in the set of equations (i.e., the edge lengths) are the parameters. If such a network runs to its limit, one or more of the parameters is changed, and then the new network is allowed to run on, will it reach a limit, supposing it has one? This question is addressed in this section.

The process of adjusting a solution to small changes in problem parameters has wide applicability in algorithm design [Frederickson 83]. Generally, it is assumed that the original solution was quite costly to obtain; therefore, updating that solution is more
reasonable than re-solving the entire problem after a small change. In program analysis, recent work has focused on incremental updates of data flow information [Reps 82, Ryder 82, Zadeck 84].

Given two networks, \( W_A \) and \( W_B \), both defined on the same bounded set \( Q \), and both monotonic, assume that \( \forall x \in Q: W_A(x) \leq W_B(x) \).

**Lemma 2.9:** If \( \forall x \in Q: W_B(x) \geq W_A(x) \) then \( \forall j \in \mathbb{N}^+: W_B^j(o) \geq W_A^j(o) \).

**Proof:**

By induction:

- \( 0 = 0 \)
- \( W_B^1(0) \geq W_A^1(0) \) (obvious)
- \( W_A(W_B^1(0)) \geq W_A^2(0) \) (condition of theorem)
- \( W_B^1(0) = W_B^1(0) \) (monotonicity)
- \( W_B^2(0) \geq W_A(W_B^1(0)) \) (obvious)
- \( W_B^2(0) \geq W_A^2(0) \) (condition of theorem)
- \( W_B^{j-1}(0) \geq W_A^{j-1}(0) \) (transitivity)
- \( W_B^{j-1}(0) \geq W_A^{j-1}(0) \) (inductive hypothesis)
- \( W_A(W_B^{j-1}(0)) \geq W_A^j(0) \) (monotonicity)
- \( W_B^j(0) = W_B^j(0) \) (obvious)
- \( W_B^j(0) \geq W_A(W_B^{j-1}(0)) \) (condition of theorem)
- \( W_B^j(0) \geq W_A^j(0) \) (transitivity)

It follows that if the network \( W_A \), which may or may not have a limit starting at \( 0 \), runs for any time, reaching a value, say \( X_A \), then the parameters are changed to get \( W_B \) which does have a finite limit and \( W_B \) is run starting at \( X_A \); it will reach its limit \( L_B \) (assuming of course that \( W_B \geq W_A \)). This shown in the next theorem.

To make the next theorem applicable to infinite as well as finite limits it is convenient to introduce at this point a uniform characterization of limits. The following lemma gives necessary and sufficient conditions for a limit, either finite or infinite, to exist.

**Lemma 2.10:** A limit \( L \) exists iff the following three conditions are satisfied:
1. \( L \) is stable

2. \( \forall j: L \geq W^i(0) \)

3. If \( X \neq L \) and \( \forall j: X \geq W^i(0) \) then \( X \geq L \).

Proof: 1 follows directly from the limit definition. 2 is true by definition for an infinite limit and follows by monotonicity for a finite limit. 3 follows from the definition for an infinite limit and is also satisfied when \( L \) is finite.

We also need another definition to make our next result applicable to infinite as well as finite limits.

For \( X \leq L \) for \( L \) stable we say \( \text{Lim}(W^i(X)) \rightarrow L \) if either of the following holds.

(This extends the notion of limit to initial non-0 values.)

1. \( \exists k \in N_0 \ni W^i(L) = L \)

2. \( \forall j \in N_0: L > W^i(X) \) and if \( Y \geq W^i(X) \forall j \in N_0 \) then \( Y \geq L \).

Similar conditions could be stated for \( X \geq L \).

**Lemma 2.11:** If \( X \) is finite for \( W_A \), then \( \text{Lim}(W^i_A(X)) \rightarrow L_A \).

**Proof:** \( L_A \geq X \geq 0 \), so \( L_A \geq W^i_A(X) \geq W^i_A(0) \), whence \( \text{Lim}(W^i_A(X)) \rightarrow L_A \).

**Theorem 2.1:** Assume \( \forall X \in Q: W_B(X) \geq W_A(X) \) and

\( X_A = L_A \), or \( W^i_A(0) \) for some \( j \), or in fact any finite value for \( W_A \).

Then \( W_B \) has a limit \( L_B \) iff \( \text{Lim}(W^i_A(X_A)) \rightarrow L_B \).

**Proof:** (\( \Rightarrow \)) Assume that \( L_B \) is the limit of \( W_B \).

If \( X_A \leq W^i_A(0) \) \( \exists z \in N_0 \) then by Lemma 2.9

\( W^i_B(0) \geq W^i_A(0) \geq X_A \), whence \( X_A \) is finite for \( W_B \).

and the result follows by Lemma 2.11.

On the other hand if \( X_A = L_A \) is an infinite limit, then

\( \forall z \in N_0: L_B \geq W^i_B(0) \geq W^i_A(0) \),

whence \( L_A \leq L_B \). Thus either \( L_A = L_B \) or \( L_A \) is finite for \( B \), and done.

(\( \Leftarrow \)) If \( \text{Lim}(W^i_A(X_A)) \rightarrow L_B \) then we can show that
the conditions of lemma 2.10 are met by \( L_B \).

First note that \( W_B^k(X_A) \geq W_A(X_A) \geq X_A \). Then:

1. \( L_B \geq X_A \) is stable by assumption.

2. \( L_B \geq X_A \geq 0 \) so
   \[ W_B = W_B = L_B \geq W_B = W_B \] by monotonicity, and finally

3. \( Vj \in N_0 \); \( Vj \in N_0 \); \( Y \geq W_A \) implies \( Vj \in N_0 \); \( Y \geq W_A \) which in turn implies
   \[ Vj \in N_0 \); \( Y \geq L_A \) by Lemma 2.10 applied to \( W_A \) and thus that
   \[ Y \geq W_A \geq L_A \), whence \( Y \geq L_B \).

Not only will \( W_B \) reach its limit, it is also easy to show that it will always reach the
limit at least as rapidly as if after the conversion it is again started with an input of \( 0 \). On
the other hand, if instead of \( W_B \geq W_A \), \( W_B \leq W_A \), no such guarantee can be given. This
is shown by the following example.

**Ex. 2.6:**

Let \( W(<x_1, x_2>) \) be the network function given by the following equations,
characterized by the operations \( n \) and \( u \), and the parameters \( A = \{1, 2, 3\}, \{2, 4\}, \{1, 3\}, \{3, 4\} \). \( Q \) is all subsets of this set. The partial
ordering is \( x \geq y \) corresponding, unintuitively, to \( x \leq y \). The minimum
element in \( Q \) is \( \{1, 2, 3, 4\} \).

\( W_A \):

\( A = \{1, 2\}, \{2, 4\}, \{1, 3\}, \{3, 4\} \).

\( x_1 = \{1, 2\} n x_1 \cup \{2, 4\} n x_2 \)

\( x_2 = \{1, 3\} n x_1 \cup \{3, 4\} n x_2 \)

Starting at \( x^0 = <\{1, 2, 3, 4\}, \{1, 2, 3, 4\}> \)

\( x^1 = <\{1, 2, 4\}, \{1, 3, 4\}> \)

\( x^2 = <\{1, 2, 4\}, \{1, 3, 4\}> = L_A \).
$W_B$:
$B = \{\{1,2,3\}, \{2,4\}, \{1,3\}, \{3,4\}\}.$
(For $X$ = any subset of $\{1,2,3,4\}$
$W_B(X) \geq W_A(X)$ and so (unintuitively)
$W_B(X) \leq W_A(X)$

$x_1 = \{1,2,3\} x_1 \cup \{2,4\} x_2$
$x_2 = \{1,3\} x_1 \cup \{3,4\} x_2$

Starting at $X^0 = \langle \{1,2,3,4\}, \{1,2,3,4\}\rangle$

$X^1 = \langle \{1,2,3,4\}, \{1,3,4\}\rangle$

$X^2 = \langle \{1,2,3,4\}, \{1,3,4\}\rangle = L_A$

However if we assumed we changed from $W_A$ to $W_B$ after the
former stabilized at $\langle \{1,2,3\}, \{1,3,4\}\rangle$ we get:

Starting at $X^0 = \langle \{1,2,3\}, \{1,3,4\}\rangle$

$X^1 = \langle \{1,2,3\}, \{1,3,4\}\rangle$ = a stable state $X_A$

$X_A = L_A$

If certain strong, though realistic, conditions hold we can always continue from a
solution of one network, $L_A$ of $W_A$, to that of a second one, $L_B$ of $W_B$, by applying $W_B$
to $L_A$, a finite number of times no matter whether $W_A \geq W_B$ or not. Such conditions are
given in the following lemma.

**Theorem 2.2:** Let $NET_A = (O,W_A)$, $NET_B = (O,W_B)$

1. $O$ is a finite set, or, more generally, if $O$ meets the ascending and
descending chain conditions

2. Both $NET_A$ and $NET_B$ have (finite) limits, $L_A$ and $L_B$ respectively.

3. $NET_A$ and $NET_B$ each have only one stable state.

Then if there is a $Y$ in $O$ which is either $\geq$ or $\leq W(Y)$, and $Y \geq X$ it follows
that $X = W_B^{-1}(Y) = L_B$.

**Proof:** Follows directly from lemma 2.5

The conditions for theorem 2.2 are satisfied by the network corresponding to the
minimum path equation set; further, they are often satisfied when the set of equations
which define the network has no cycle in its dependency graph. On the other hand, if a
set of equations represents a network which has such a cycle, one may be able to partially
solve it symbolically in terms of its parameters. We then obtain a set of equations with
no such cycles and therefore one with a unique solution and only one vector of variable
values for which it is stable.
REFERENCES

Design And Analysis Of Computer Programs.

The Theory of Parsing, Translation, and Compiling.

[Birkhoff 67] Birkhoff, G.
Lattice Theory.

[Cocke 70] Cocke, J.
Global Common Subexpression Elimination.
In Proceedings of ACM SIGPLAN Symposium on Compiler

[Fong 75] Fong, A. C., Kam, J. B., and Ullman, J. D.
Applications of Lattice Algebra to Loop Optimization.
In Conference Record of the ACM Symposium on the Principles of
Programming Languages, pages 1–9. Association for Computing

Data structures for on-line updating of minimum spanning trees.
In STOC, pages 252–257. ACM-SIGACT, April, 1983.

[Hecht 77] Hecht, M. S.
Flow Analysis of Computer Programs.

[Kam 76] Kam, J. B. and Ullman, J. D.

[Kildall 73] Kildall, G.
A Unified Approach to Global Program Optimization.
In Conference Record of the ACM Symposium on the Principles of
Programming Languages, pages 194–206. Association for

[Reps 82] Reps, T.
Optimal-time Incremental Semantic Analysis for Syntax-directed Editors.
In Conference Record of the Ninth Annual ACM Symposium on
Principles of Programming Languages, pages 169–176.
Association for Computing Machinery-SIGPLAN, January, 1982.
[Ryder 82] Ryder, B. G.
Incremental Data Flow Analysis.
In Conference Record of the Tenth Annual ACM Symposium on
Principles of Programming Languages, pages 167-176.
Association for Computing Machinery-SIGPLAN, January, 1982.

[Zadeck 84] Zadeck, F. K.
Incremental Data Flow Analysis in a Structured Program Editor.
In Proceedings of SIGPLAN '84 Symposium on Compiler Construction,
INDEX

Ascending chain condition 10

Bounded function 6
Bounded set 6

Descending chain condition 10

Finite Limit 8
Finite state 7
Fixed point 5, 7

Infinite Limit 8
Infinite state 7

Limit 8

Monotonic 6

Network 7

Seeks 8
Stable 7

$W^n(X)$ 8