HIGH ORDER NUMERICAL
SOMMERFELD BOUNDARY CONDITIONS:
THEORY AND EXPERIMENTS

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HIGH ORDER NUMERICAL SOMMERFELD BOUNDARY CONDITIONS:
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We develop high-order, non-reflecting boundary equations for a semi-discrete approximation of the simple (hyperbolic) advection equation \( U_t + cU_x = 0 \). These boundary equations are based on a discrete interpretation of Sommerfeld's radiation condition for a second order wave equation which is associated with the semi-discrete equations. The performance of these schemes is expressed by an exact measure of the energy reflected at the boundary. For low order cases, the discrete Sommerfeld boundary equations are identical with the standard finite difference equations, but for high orders of approximation (starting with 4 points), the discrete Sommerfeld schemes differ from standard finite differences. It is shown, and verified experimentally, that the discrete Sommerfeld schemes are optimal, in the sense that they produce the least amount of reflected energy.

Moreover, it is known theoretically, and we verify experimentally, that the reflected energy remains invariant when the semi-discrete equations are time-discretized with the trapezoidal (Crank-Nicolson) method. The corresponding fully discrete boundary equations are thus also optimal in the sense that they minimize the reflected energy.
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1. INTRODUCTION

Consider the simple hyperbolic equation
\[
\frac{\partial U}{\partial t} + c \frac{\partial U}{\partial x} = 0
\]
(1)
on the semi-infinite domain \(x \in \mathbb{D} = (-\infty, 0]\). Numerical approximations with centered
discretizations require that a boundary condition be specified at the outlet point \(x = 0\).
This condition is of course spurious, required only by the approximating process, since no
such condition is required by the original equation. In this paper we show that this
boundary condition may be considered a Sommerfeld, non-reflecting condition for a second
order hyperbolic equation associated with the numerical approximation of (1). We also
show that a correct implementation of this non-reflecting condition is obtained by
considering non-reflection as a discrete, numerical condition rather than as a continuous,
analytical condition. While the two viewpoints produce the same results for low order
approximations, the results differ when high order non-reflecting schemes are sought. For
these schemes, the discrete interpretation of Sommerfeld's condition produces results which
are superior to the continuous interpretation (meaning that the reflected energy is less).
This is demonstrated theoretically and verified by numerical experiments. It is in the
derivation, and analysis with energy methods, of these superior, high order absorbing
boundary conditions that the principal new contribution of this paper lies.

The simple three point semi-discretization of (1):
\[
\frac{dU_j}{dt} = -c \left( \frac{U_{j+m} - U_{j-m}}{2h} \right) \cdot A \cdot U_j
\]
(2)
is a convenient model since it will give the essential results without unnecessary
complications. While this model is semi-discrete, it has been proven elsewhere [7], [8]
that the expression for the reflected energy remains unchanged when discrete time
integration is performed by the trapezoidal (Crank Nicolson) method. Thus, the results and
conclusions derived with the semi-discretization (2) are identically applicable to the
 corresponding full discretizations. This is also verified experimentally.

2. SOMMERFELD CONDITIONS FOR THE WAVE EQUATION

We briefly depart from the present problem to analyze a related question. Consider
Cauchy's problem for the wave equation:

$$\frac{\partial^2 U}{\partial t^2} - c^2 \frac{\partial^2 U}{\partial x^2} = 0$$

(3)

on the entire real axis with far boundary conditions:

$$U(x, t) = 0$$

(4)

It may be desirable, for obvious analytical or computational reasons, to reduce the domain
of the equation to some finite length, say \(x \in [-\ell, 0]\). Boundary conditions are now
required; they may be derived by the following argument: (we consider the point \(x=0\)
only; the results apply equally in \(x=\ell\)).

The rewriting of (3) as

$$\left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) U = 0$$

(5)

reveals the existence of two types of solutions, viz:

(i) Rightgoing solutions which satisfy

$$\frac{\partial P(x, t)}{\partial t} + c \frac{\partial P(x, t)}{\partial x} = 0$$

(6)

and,

(ii) Leftgoing solutions which satisfy

$$\frac{\partial Q(x, t)}{\partial t} - c \frac{\partial Q(x, t)}{\partial x} = 0$$

(7)
Any \( U(x,t) \) may be expressed as the sum of two such solutions:
\[
U(x,t) = P(x,t) + Q(x,t)
\]
(8)

When a fictitious boundary was introduced in \( x=0 \), it was implicitly assumed that the remaining space \( x>0 \) was inert, i.e., that no leftgoing solution could exist in \( x=0 \). The condition which expresses this may be written as
\[
Q(0, t) = 0
\]
(9)

or, by (6) and (8):
\[
\left( \frac{3U}{2t} + c \frac{\partial U}{\partial x} \right) = 0
\]
(10)

The latter is called a Sommerfeld radiation condition, (see e.g. Courant and Hilbert, (1962) Vol. II), and may be used as the extra boundary condition needed in the fictitious point \( x=0 \).

3. DOWNSTREAM BOUNDARY CONDITION FOR A FIRST ORDER HYPERBOLIC EQUATION

What similarity the numerical treatment of a downstream boundary condition of the first order hyperbolic equation (1) has with Sommerfeld's radiation condition will now be explained. It is known that the three point semi-discretization (2) is a consistent approximation of the second order wave equation (3). To show this, relabel \( U_j \) as \( u_j \) for \( j \) odd in (2). This results in the system of semi-discrete equations
\[
\frac{du_j}{dt} = -c \frac{u_{j+1} - u_{j-1}}{2h}, \quad \frac{dv_j}{dt} = -c \frac{u_{j+2} - u_{j}}{2h}
\]
(11)

which becomes, when \( h \rightarrow 0 \), a consistent approximation of
\[
\frac{\partial U}{\partial t} = -c \frac{\partial V}{\partial x}, \quad \frac{\partial V}{\partial t} = -c \frac{\partial U}{\partial x}
\]
(12)

Elimination of \( V \) then results in (3), q.e.d. ■
The boundary in $X = 0$ may thus be treated as a boundary condition for the wave equation (3), specifying that no solution of $Q$ type may exist in that point. The corresponding Sommerfeld condition (10) is, not so unexpectedly, the expression of the original equation (1) itself.

4. NUMERICAL BOUNDARY CONDITIONS

A classical procedure for handling the boundary numerically consists in approximating the analytic Sommerfeld condition (10) by usual finite difference procedures. The simplest approximation is the familiar two point formula:

$$\frac{dU_0}{dt} = -c \left( \frac{U_0 - U_{-1}}{h} \right)$$

(13)

which has the truncation error ($U$ is a genuine solution of (1)):

$$T_h = \frac{2U(0,t)}{\partial t} + c \left( \frac{U(0,t) - U(-h,t)}{h} \right) = - \frac{c h U''}{2} + \ldots = O(h)$$

(14)

More accurate approximations use more than two points, with a corresponding increase in the order of $T_h$. By contrast, one may attempt to satisfy (9) as well as possible, instead of (10) as well as possible. This method minimizes reflection in the numerical process described by (2), and is thus consistent with the discrete model of the equation. We will henceforth refer to this as the consistent discrete-Sommerfeld method.

The solution in $D$ consists of the solution that would be obtained if the boundary were not present, plus a solution which is reflected at the numerical boundary back into $D$. We shall give below exact analytic expression for the energy which is contained in this reflected solution, and use this energy measure to compare the merits of different schemes. It will then be found that, as expected, the high order discrete Sommerfeld schemes are superior to the classical finite difference boundary schemes.
5. **SYNTHESIS**

The consistent discrete Sommerfeld procedure consists in attempting to satisfy the numerical counterpart of (9) as well as possible. To that end, we consider a Fourier analysis which yields a division of numerical solutions into their rightgoing and leftgoing components. It has been shown in [16] that when solutions of (2) are expressed by their \( \mathrm{i}- \)Fourier transforms:

\[
\hat{u}_j(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_j(t) e^{-\mathrm{i} \omega t} \, dt
\]

then the numerical images of (6) and (7) satisfy the relations (with \( \omega = \omega/\kappa/c \))

\[
\hat{p}_j^{\mathrm{r}} = \hat{e}_j(\omega) = -\mathrm{i} \omega + \sqrt{1 - \omega^2}
\]

and

\[
\hat{q}_j^{\mathrm{r}} = \hat{e}_j(\omega) = -\mathrm{i} \omega - \sqrt{1 - \omega^2}
\]

respectively. Each consists (for each \( \omega \)) of sinusoidal wave-like solutions. Those of \( \{ p_j \} \) type have a positive group velocity and carry energy from left to right, while solutions of \( \{ q_j \} \) type have a negative group velocity and carry energy from right to left.

The specification that no leftgoing solution be present in \( x=0 \) results in

\[
\hat{q}_0 = \hat{e}_2(\omega) \hat{q}_{-1} = 0
\]

thus giving the consistent discrete Sommerfeld condition:

\[
\hat{u}_0(\omega) = \hat{e}_1(\omega) \cdot \hat{u}_\infty(\omega)
\]

As such, this condition cannot be implemented as an equivalent differential equation because \( \hat{e}_1(\omega) \) is not rational in \( \omega \). But rational approximations of increasing order of accuracy provide workable boundary conditions which produce increasingly small amounts of reflection.
6. REFLECTION RATIOS

A measure of the efficiency of boundary schemes is obtained with ratios of Fourier transforms. Consider a solution of \{p_j\} type which arrives at the boundary, and the resulting reflected solution of \{q_j\} type. The **amplitude reflection ratio** at the boundary is defined as:

$$
\gamma(\omega) = \frac{\hat{q}_r(\omega)}{\hat{p}_r(\omega)}
$$

(19)

which ideally should be identically zero. The expression is of \(\gamma(\omega)\) is obtained by taking the Fourier transform of the boundary scheme and then using the expressions (16) – (17) to relate \(\hat{p}_r, \hat{q}_r, \ldots\) etc. to \(\hat{p}_o, \hat{q}_o\). For the two-point formula (2) we find:

$$
i\omega (\hat{p}_o + \hat{q}_o) = -\frac{E}{h} (\hat{p}_o + \hat{q}_o - \hat{E}_r^r \hat{p}_o - \hat{E}_r^q \hat{q}_o)
$$

(20)

From which we derive

$$
\gamma(\omega) = -\frac{i\omega}{i\omega + \frac{1 + \hat{E}_r^r}{1 - \hat{E}_r^q}} = -\frac{1 - \sqrt{1 - \hat{E}_r^q}}{1 + \sqrt{1 - \hat{E}_r^q}}
$$

(21)

7. REFLECTED ENERGY: Analytical Expression

The energy of \(\{U_j\}\) on \(D = [-\omega, 0]\) is defined as

$$
\mathcal{E}(t) = \|U_j(t)\|_2^2 = h \sum_{j < 0} |U_j|^2
$$

(22)

\(\|U_j\|_2^2\) is also the square of the \(\ell^2\) norm of \(\{U_j\}\) and may be expressed in Fourier space by Parseval's relation as

$$
\|U_j(t)\|_2^2 = \frac{\sum_{\xi} \mathcal{U}(\xi, t) \xi^2 d\xi}{\sqrt{2\pi}}
$$

(23)

where \(\mathcal{U}(\xi, t)\) is the discrete Fourier transform of \(\{U\} :\)

$$
\mathcal{U}(\xi, t) = h \sum_{j < 0} U_j(t) e^{-i\xi j h}
$$

(24)
Moreover,
\[ \mathcal{U}(\xi, t) = \mathcal{P}(\xi, t) \quad \text{when} \quad |\xi| < \frac{\pi}{2h} \]  
(25)

and
\[ \mathcal{U}(\xi, t) = \mathcal{Q}^-(\xi, t) \quad \text{when} \quad \frac{\pi}{2h} \leq |\xi| \leq \frac{\pi}{h} \]  
(26)

with a corresponding separation of energy. When reflected at a boundary, energy which is in an incident solution of \( \{\rho\} \) type is partially reflected as energy in a solution of \( \{\sigma\} \) type.

The energy reflected at the boundary is directly related to the reflection ratio \( \mathcal{R}(\omega) \) (and not the truncation error at the boundary). To show this, consider an initial \( \{\nu_j(0)\} \) which is mostly of \( \{\rho\} \) type, i.e.

\[ \mathcal{U}(\xi, 0) = 0 \quad \text{when} \quad \frac{\pi}{2h} \leq |\xi| \]  
(27)

For sufficiently large \( t \), it will have passed entirely through the boundary \( x = 0 \):

\[ \lim_{t \to \infty} \{u_j(t)\} = \{q_j(t)\} \]  
(28)

The reflected energy may be expressed as (from [14])

\[ E_R = \lim_{t \to \infty} \int_{-\frac{\pi}{2h}}^{\frac{\pi}{2h}} |\mathcal{Q}^-(\xi, t)|^2 \frac{d\xi}{2\pi} \]  
(29)

The expression of \( \mathcal{Q}(\xi) \) (instead of \( \mathcal{P}(\omega) \)) which is needed in (29) is obtained by means of the dispersion relation of the semi-discretization (2):

\[ \omega = \frac{\xi}{h} \sin(\frac{\xi \cdot h}{2}) \]  
(30)

E.g., for the two-point boundary condition (13), the expression for the reflection ratio becomes

\[ \mathcal{R}(\xi) = -\frac{1 - \sqrt{1 - \sin^2(\xi h)}}{1 + \sqrt{1 - \sin^2(\xi h)}} = -\frac{1 - \cos(\xi h)}{1 + \cos(\xi h)} \]  
(31)
It can be verified algebraically that the rate of convergence to zero, when \( h \to 0 \), of the \( \mathcal{L} \) norm of the reflected solution (which is the square root of the reflected energy) is the same as that of \( \mathcal{E}(\varepsilon) \) when \( \varepsilon h \to 0 \), which is the same as that of \( \mathcal{F}(\omega) \) when \( \omega h \to 0 \).

8. NON-REFLECTING SCHEMES

Boundary equations which are less reflecting than (13) may be derived by a method of undetermined coefficients. This method consists in writing a generalized equation with a finite number of terms:

\[
\frac{dU_o}{dt} = a U_o + b U_i, + c U_{-2} + \cdots
\]

(32)

deriving the expression for the corresponding reflection ratio,

\[
\mathcal{F} = \frac{i \omega - a - b \tilde{E}_1 - c \tilde{E}_2^2}{i \omega - a - b \tilde{E}_2 - c \tilde{E}_2^2} \cdots
\]

(33)

then in finding the coefficients \( (a, b, c, \ldots) \) which maximize the order of \( \mathcal{F}(\omega) \) when \( \omega h \to 0 \). This technique produces the following results:

Three point consistent discrete Sommerfeld:

\[
\frac{dU_o}{dt} = -\frac{\varepsilon}{2h} (3U_o - 4U_i + U_{-2})
\]

(34)

Four point consistent discrete Sommerfeld:

\[
\frac{dU_o}{dt} = -\frac{\varepsilon}{2h} (4U_o - 7U_i + 4U_{-2} - U_{-4})
\]

(35)

While (34) is identical to the corresponding formula obtained with finite differences, (35) is not.

The expression of reflection ratios and truncation errors for the 2, 3, and the two 4 point formulae have been included in Table 1.

A comparison of the classical and discrete Sommerfeld methods shows that:
1. the two and three point classical and discrete Sommerfeld schemes are identical
2. the four point classical and four point discrete Sommerfeld schemes are different, with the latter resulting, as expected, in a better reflection ratio but a worse truncation error than the former.

Since the discrete Sommerfeld schemes give the highest possible order for the reflection ratio, it is easily proven by use of (29) that they are optimal in the sense that
FIGURE 1: Reflection ratio $\rho(\omega)$ for the 2 point, 3 point, 4 point finite differences and 4 point Sommerfeld boundary equations. It may be observed that $\rho_{4s} \leq \rho_{4f}$ for all $\omega \in (0, c/\lambda)$. Thus, by (29) – (30), the energy reflected by the 4 point Sommerfeld equation is no greater than that reflected by the 4-point finite differences equation in all possible cases.
<table>
<thead>
<tr>
<th></th>
<th>2 POINT</th>
<th>3 POINT</th>
<th>4 POINT FINITE DIFFERENCES</th>
<th>4 POINT DISCRETE SOMMERFELD</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>BOUNDARY EQUATION</strong></td>
<td>$\frac{dU_0}{dt} = B \cdot U_0 = -\frac{C}{h}(U_2 - U_1)$</td>
<td>$-\frac{C}{2h}(3U_0 - 4U_1 + U_2)$</td>
<td>$-\frac{C}{6h}(14U_0 - 18U_1 + 9U_2 - 2U_3)$</td>
<td>$-\frac{C}{8h}(4U_0 - 7U_1 + 4U_2 - U_3)$</td>
</tr>
<tr>
<td><strong>TRUNCATION ERROR</strong></td>
<td>$\frac{dU}{dx} - B \cdot U = 0(h^2)$</td>
<td>$-\frac{C}{3}U'' = 0(h^2)$</td>
<td>$-\frac{C}{4}U'' = 0(h^2)$</td>
<td>$-\frac{C}{6}U'' = 0(h^2)$</td>
</tr>
<tr>
<td><strong>REFLECTION RATIO</strong></td>
<td>$\phi = \frac{\hat{G}_0}{\hat{B}_0}$</td>
<td>$\frac{i\omega + 1}{i\omega + 1 - \hat{E}_1^{-2}}$</td>
<td>$\frac{i\omega + 1}{i\omega + 1 - \hat{E}_1^{-2}}$</td>
<td>$\frac{i\omega + 1}{i\omega + 1 - \hat{E}_1^{-2}}$</td>
</tr>
<tr>
<td></td>
<td>$= \frac{i\omega + 1}{i\omega + 1 - \hat{E}_1^{-2}}$</td>
<td>$+ \frac{i\omega + 1}{i\omega + 1 - \hat{E}_1^{-2}}$</td>
<td>$= \frac{i\omega + 1}{i\omega + 1 - \hat{E}_1^{-2}}$</td>
<td>$= \frac{i\omega + 1}{i\omega + 1 - \hat{E}_1^{-2}}$</td>
</tr>
<tr>
<td></td>
<td>$= 0[(\omega h)^3]$</td>
<td>$= 0[(\omega h)^3]$</td>
<td>$= 0[(\omega h)^3]$</td>
<td>$= 0[(\omega h)^3]$</td>
</tr>
</tbody>
</table>

**TABLE 1**: Boundary equations, the corresponding truncation errors and reflection ratios.
when \( n \to 0 \), the energy reflected at the boundary for a given \( \mathcal{U}(x,0) \) in \( D \) is the least for all the schemes of the general form (23). Moreover, we observe (Figure 1) that \( f_4 < f_5 \) for all \( \omega \). Thus, by (29)-(30), the energy reflected at the boundary by the 4-point Sommerfeld equation is no greater than that reflected by the 4-point finite difference equation in all possible cases, even when \( \lambda \) is finite, not going to zero. See Table 3 for a numerical illustration of this property.

Note that the method of undetermined coefficients used here consists of seeking to approximate \( \rho = 0 \) with the irrational function (34). Another approach would consist in approximating \( E_j(\omega) \) with a rational (Taylor or Padé) expansion to be substituted in (19). Arguments of this kind have been used by Halpern (1982) and by Engquist and Majda (1977) to develop absorbing boundaries for the second order wave equation, respectively for the one dimensional (discrete) and two dimensional (analytic) cases.

9. FULL DISCRETIZATIONS

We now invoke an important result which is proven in elsewhere. Consider the full discretizations obtained when the semi-discretization (2) and boundary equation (13) are integrated in time by an energy conservative time marching method.

The Crank–Nicolson (or trapezoidal) method:

\[
\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{1}{2} \left( A \cdot U_j^{n+1} + A \cdot U_j^n \right)
\]

which we shall use later on is an example of an energy conservative method.

Then (14) the reflected energy of these full discretizations remains strictly equal to that of the semi-discrete method (2)-(13) and is still expressed by (29).

A numerical verification of this invariance is given below in Figure 4 and Table 3.

The importance of this result in the present study is as follows: since the semi-discrete boundary equations of Section 8 are optimal in the sense that they minimize the reflected energy, and since the reflected energy is invariant under time-discretization with an energy conservative method such as (36), it follows that the corresponding full discretizations remain optimal in the same sense.
10. **STABILITY**

We have not carried out a formal analysis of the stability of the semi- and full discretizations which correspond to the non-reflecting schemes described in this paper. There is, however, theoretical evidence (by consideration of the reflection ratios illustrated in Figure 1) and numerical evidence (from the experimental results described below) that stability exists in all cases. By contrast, if the leapfrog method of time marching

\[
\frac{U_j^{n+1} - U_j^{n-1}}{2\Delta t} = A \cdot U_j^n
\]

(37)

were used instead of the trapezoidal method, then unstable parasitic modes in \( D \) would be supported by the boundary schemes. (see Trefethen [10] for a description of appropriate tools for an analysis of this question).
11. REFLECTED ENERGY: Numerical Experiments

A series of numerical experiments have been conducted with the objective of verifying the theoretical results. A Gaussian initial function

\[ U(x,0) = e^{-\frac{1}{2}[(x-x_o)/\sigma]^2} \quad x_o = -50 \quad \sigma = 10 \]  

was prescribed on the domain \( D = [-100,0] \) for the semi-discretization (2) with \( n=1 \) and \( n=2 \). The Fourier transform of (38) is:

\[ \hat{U}(\xi,0) = \int_{-\infty}^{\infty} U(x,0) e^{-i \xi x} dx = \sigma \sqrt{2 \pi} e^{-\frac{1}{2} \xi^2 / \sigma^2} \]  

The energy of the Gaussian which lies outside of \( D \) is less than \( 10^{-7} \) times the total energy, and with \( \sigma/h = 5 \) or 10, the energy of \( \hat{U}(\xi,0) \) which is outside of the \( \{ \rho \} \) band \( |\xi h| \leq \pi/2 \) is less than \( 10^{-11} \) times the total energy. The asymptotic approximation

\[ \hat{Q}(\xi,0) \approx 0 \quad (40) \]
\[ \hat{F}(\xi,0) \approx \hat{U}(\xi,0) \]  

is thus fully justified.

The initial function (38) may be considered (to within numerical accuracy) as a wave packet of finite support in \( x \), with wave number \( \xi h \to 0 \). The reflected solution is also a wave packet of finite support in \( x \), with wave number \( \xi h \to \Pi_p \) or wave length \( \lambda \to 2h \) [15]. The sawtootheed nature of this reflected solution is illustrated in Figure 2.

Quantitative aspects of these numerical experiments which verify the theoretical results of this paper are given in Figures 3 and 4, and in Tables 2 and 3.

The energy-versus-time plot in Figure 2 verifies the theory that reflection is completed in a finite time (to within numerical accuracy). In Table 2, convergence rates (obtained by dividing \( h \) by 2) are compared with analytic predictions. And the experiments reported in Figure 2 and Table 3 verify the invariance property described in Section 9.
FIGURE 2: Illustration of reflection in $x = 0$, obtained by numerical integration of (2) with the 2 point boundary equation (13) and a Gaussian initial function. Time marching is performed with the Crank–Nicolson method and a Courant number $R = \frac{ca^t}{h} = 0.1$. 
FIGURE 3: Energy in D as a function of time for the semi-discretization (2), the 2 point boundary equation (13) integrated in time with the Crank-Nicolson method with a Courant number $R = 0.1$, and initial condition (38) (this is the same case as that illustrated in Figure 2) and corresponds to the second column of Table 2. The reflected energy $\mathcal{E}_r = 1.3750963 \times 10^{-3}$ measured in this experiment verifies the result obtained by numerical quadrature of the integral (29) to within arithmetical accuracy.
FIGURE 4: Numerical verification of the invariance of the reflected energy to time discretization, described in Section 9. The experiment of Figure 3 was repeated with constant $h$ and variable $\Delta t$, corresponding to Courant numbers $R$ equal to 0.1, 0.5, 1.0, 2.0, 3.0, 4.0, 5.0, 6.0, and 7.0. While the nature of the solution is affected by the increase in $R$ (or $\Delta t$), the total energy reflected (obtained when $t \to \infty$) is indeed observed to be invariant. (A complete proof of this property may be found in [14]). See also Table 3.
<table>
<thead>
<tr>
<th>Boundary Scheme</th>
<th>Order of ( \rho )</th>
<th>energy ratio (asymptotic)</th>
<th>Reflected Energy = ( h \sum \frac{u_j}{2} )</th>
<th>energy ratio (actual)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( m )</td>
<td>( \left( \frac{h_1}{h_2} \right)^{2m-1} = 2^{2m} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 point</td>
<td>2</td>
<td>16</td>
<td>( h_1 = 2 )</td>
<td>16.412</td>
</tr>
<tr>
<td>3 point</td>
<td>3</td>
<td>64</td>
<td>( h_1 = 2 )</td>
<td>67.544</td>
</tr>
<tr>
<td>4 point (F.D.)</td>
<td>3</td>
<td>64</td>
<td>( h_1 = 2 )</td>
<td>106.110</td>
</tr>
<tr>
<td>4 point (Sommerfeld)</td>
<td>4</td>
<td>256</td>
<td>( h_1 = 2 )</td>
<td>280.978</td>
</tr>
</tbody>
</table>

TABLE 2: Convergence tests for the boundary equations. These values have been obtained by repeating a numerical experiment similar to that illustrated in Figure 2 with two values of \( h \). In each case, the actual value of the reflected energy was also verified to be given by the integral (29) (evaluated by numerical quadrature). When \( h \) becomes small, than the numbers in column 3 (asymptotic convergence rate) and those in column 6 (observed convergence rate when \( h_1 \) is divided by 2) become identical.
TABLE 3: Illustration of the invariance of the reflected energy to time-discretization. These numbers, which are the same as those used to obtain Figure 4, have been obtained by numerical integration of (2)-(13) with $h=1$ and variable $\alpha_1$ or $R$. (see Figure 4). The final value $E_R^{\infty}=1.3750\times10^{-1}$ is also obtained to within arithmetical accuracy by numerical quadrature of (29)-(39).
References


