SINGULAR INTEGRAL EQUATIONS - THE CONVERGENCE
OF THE NYSTRÖM INTERPOLANT OF THE
GAUSS-CHEBYSHEV METHOD

by

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ABSTRACT

Nyström's interpolation formula is applied to the numerical solution
of singular integral equations. For the Gauss-Chebyshev method, it is
shown that this approximation converges uniformly, provided that the
kernel and the input functions possess a continuous derivative.
Moreover, the error of the Nyström interpolant is bounded from above
by the Gaussian quadrature errors and thus converges fast, especially
for smooth functions. For C^∞ input functions, a sharp upper bound for
the error is obtained. Finally numerical examples are considered.
It is found that the actual computational error agrees well with the
theoretical derived bounds.

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Research Council.
1. **Introduction** In this paper we consider the convergence of the natural or Nyström interpolant of the direct Gauss-Chebyshev method, for the numerical solution of Cauchy-type singular integral equations of the form:

$$\pi^{-1} \int_{-1}^{1} (1-t^2)^{-1/2}(t-s)^{-1} y(t) \, dt + \lambda \int_{-1}^{1} (1-t^2)^{-1/2} K(s,t)y(t) \, dt = f(s), \quad -1 < s < 1$$

$$\pi^{-1} \int_{-1}^{1} (1-t^2)^{-1/2} y(t) \, dt = N,$$

where $K(s,t)$, $f(s)$ are given input functions and $N$ is a known constant. If we assume that $K(s,t)$, $f(s)$ are Hölder-continuous functions in $[-1,1] \times [-1,1]$ and $[-1,1]$ respectively, then it is well known [8] that (1.1), (1.2) possess a unique solution $y(t)$ in the space of Hölder-continuous functions.

Erdogan and Gupta [2] proposed a direct method for the solution of (1.1), (1.2), which is based on a quadrature approximation of the integrals. In particular, if the Gauss-Chebyshev quadrature formula is applied on a certain set of points, then (1.1), (1.2) can be reduced to an algebraic system of the form,

$$\begin{align*}
(A_n + \lambda C_n) y_n^* &= f \\
\end{align*}$$

where

$$y_n^* = [y_n^*(t_1), \ldots, y_n^*(t_n)]^T, \quad f = [f(s_1), \ldots, f(s_{n-1}), N]^T$$

$$\begin{align*}
(A_n)_{i,j} &= n^{-1} (t_j - s_i)^{-1}, \quad (A_n)_{n,j} = n^{-1} \\
(C_n)_{i,j} &= \pi n^{-1} K(s_i, t_j), \quad (C_n)_{n,j} = 0,
\end{align*}$$
and \( t_j = \cos[(2j-1)\pi / 2n] \) \( s_j = \cos[i\pi/n] \), for all \( i=1(1)n-1, j=1(1)n \).

The solution of the linear algebraic system (1.3), is an approximation of the solution of (1.1), (1.2) at a discrete set of points \( t_j \). Since the solution of (1.1) at points different than \( t_j \) often represent quantities of great interest in engineering, (e.g. \( y(\pm 1) \) represent the stress intensity factor), an interpolation formula is required. Erdogan and Gupta [2] have suggested a quadratic "extrapolation" technique for the evaluation of \( y(\pm 1) \). Similarly Krenk [7] has introduced a summation formula which is based on the Lagrange interpolating polynomials \( L_n(t) \) at \( (t_j,y_n(t_j)) \). Ioakimidis and Theocaris [6] and Tsamasphyros and Theocaris [12], have considered the convergence of \( L_n(t) \), and have shown that \( L_n(t) \) will converge to \( y(t) \), provided that \( K(s,t) \) and \( f(s) \) possess a continuous derivative. We note that the Lagrange interpolation formula is exact for polynomials of degree less or equal to \( n \), whereas the Gaussian quadrature that approximates (1.1) and (1.2) is exact for polynomials of degree less or equal to \( 2n \) and \( 2n-1 \) respectively. Consequently, the use of Lagrange polynomials will result in a significant loss of accuracy when compared to the use of an interpolation formula which is exact for polynomials of higher degree, say \( 2n \). Until recently [11], [5] the natural or Nyström interpolation formula has been completely ignored, although it is well known [1] that for Fredholm integral equations it can yield excellent results. Although some equivalence results for the Nyström interpolant of the direct and indirect Gauss-Chebyshev method have been given [5], the question of convergence and computational efficiency of the Nyström interpolant has not been studied.

In section 2 we introduce the Nyström interpolation formula and give a brief description of some equivalence and existence results of the discrete direct and indirect Gauss-Chebyshev method described in [4].

In section 3, we extend the analysis of [4], and use Nyström's theory [1], to show that, if \( \lambda \) is not an eigenvalue of (1.1), (1.2) then:
(i) The algebraic system (1.3) possesses a unique solution for sufficiently
large $n$.

(ii) The Nyström interpolant converges uniformly to the solution
of (1.1), (1.2), provided that $K(s,t) \in C([-1,1] \times [-1,1])$, $f(s) \in C([-1,1])$.

(iii) The error of the Nyström interpolant is bounded above by Gaussian quadrature
errors and thus it is a fast convergent interpolation formula, especially
for smooth functions.

In section 4 we solve three integral equations which arise
in the solution of Elasticity problems, and compare the convergence of the
"quadratic" extrapolation technique [2], Krenk's [7] summation formula, and
the Lobatto-Chebyshev method [10], with the Nyström interpolant. For these
eamples, we observe that the Nyström interpolant converges at least
as fast or even faster than all of the previously mentioned methods.
Finally, for functions $f \in C([-1,1])$, a sharp upper bound for the error is derived.

2. Regularized Equations

Using the Carleman-Vecua method of reduction [8], equation (1.1), (1.2)
is reduced into an equivalent Fredholm Integral equation:

\[(2.1)\quad y(t) + \lambda \pi^{-1} \int_{-1}^{1} (1-x^2)^{-1/2} L(x,t)y(x)dx = F(t)\]

where

\[(2.2)\quad F(t) = -\pi^{-1} \int_{-1}^{1} (1-s^2)^{-1/2} (s-t)^{-1} f(s)ds + N\]
\[ (2.3) \quad L(x,t) = - \int_{-1}^{1} (1-s^2)^{1/2}(s-t)^{-1} k(s,x) ds. \]

Let us assume that \( L(x,t) \in C([-1,1] \times [-1,1]) \), \( F(t) \in C([-1,1]) \) and approximate the integral part of (2.1) using the Gauss-Chebyshev quadrature. Then (2.1) is reduced to a functional equation of the form:

\[ (2.4) \quad y_n(t) + \lambda n^{-1} \sum_{m=1}^{n} L(t_m, t) y_n(t_m) = F(t) \]

where \( t_m = \cos \left( \frac{(2m-1)\pi}{2n} \right), m=1(1)n \). Furthermore, if we set \( t=t_i, i=1(1)n \), in (2.4) we obtain the algebraic system

\[ (2.5) \quad (I+\lambda Q_n) \mathbf{z} = \mathbf{F} \]

where

\[ (2.6) \quad (Q_n)_{i,j} = n^{-1} L(t_j, t_i) \quad i,j = 1(1)n \]

and \( \mathbf{z} = [z_1, \ldots, z_n]^T \), \( \mathbf{F} = [F(t_1), \ldots, F(t_n)]^T \).

We can see that there exists a unique correspondence between the solution \( z_i \) of (2.5) and the solution \( y_n(t_i) \) of (2.4). This implies that if (2.4) possesses a solution then (2.5) possesses a solution and vice versa (for more details see [1], p. 88).

For the remainder of this paper we will assume that \( K(s,x) \in C^1([-1,1] \times [-1,1]) \) and \( f(s) \in C([-1,1]) \). If we define \( g(t,x,s) \) and \( h(t,s) \) by

\[ (2.7) \quad g(x,t,s) = \begin{cases} \frac{K(s,x) - K(t,x)}{(s-t)} & \text{if } s\neq t \\ \frac{K(s,x)}{s} & \text{if } s= t \end{cases} \]
2.8) \( h(t,s) = \begin{cases} \frac{[f(s)-f(t)]}{(s-t)} & \text{if } s\neq t \\ f'(s) & \text{if } s=t \end{cases} \)

then \( g(t,x,s) \in C([-1,1] \times [-1,1] \times [-1,1]) \) and \( h(t,s) \in C([-1,1] \times [-1,1]) \).

We approximate \( F(t) \) and \( L(x,t) \) by

2.9) \( F(t) = F_n(t) + r_n(f;t) \)

2.10) \( L(x,t) = L_n(x,t) + r_n(k;x,t) \)

where

2.11) \( F_n(t) = -n^{\frac{1}{2}} \sum_{k=1}^{n-1} (1-s_k^2)(s_k-t)^{-1}f(s_k)+f(t)T_n(t)/U_{n-1}(t) + N \)

2.12) \( L_n(x,t) = -n^{\frac{1}{2}} \pi \sum_{k=1}^{n-1} (1-s_k^2)(s_k-t)^{-1}k(s_k,x)+\pi k(t,x)T_n(t)/U_{n-1}(t) \)

2.13) \( r_n(f;t) = -\pi^{-\frac{1}{2}} \int_{-1}^{1} (1-s^2)^{1/2} h(t,s) ds - n^{1/2} \sum_{k=1}^{n-1} (1-s_k^2) h(t,s_k) \]

2.14) \( r_n(k;x,t) = -\pi^{-\frac{1}{2}} \int_{-1}^{1} (1-s^2)^{1/2} g(x,t,s) ds - n^{1/2} \sum_{k=1}^{n-1} (1-s_k^2) g(x,t,s_k) \]

and \( T_n(t), U_{n-1}(t) \) are the Chebyshev polynomials of the first and second kind respectively.

We substitute \( L(t_m,t) \) with \( L_n(t_m,t) \), and \( F(t) \) with \( F_n(t) \) in (2.4) to obtain a new functional equation:
\[(2.15) \quad y_n^*(t) + \chi n^{-1} \sum_{m=1}^{\frac{n}{\chi}} L_n(t_m, t) y_n^*(t_m) = F_n(t), \]

which can be reduced to the following linear algebraic system

\[(2.16) \quad (I + \chi \bar{Q}_n) y_n^* = F_n \]

where

\[(2.17) \quad (\bar{Q}_n)_{i,j} = n^{-1} L_n(t_j, t_i), \quad i,j = 1(1)n \]

and \(\bar{F}_n = [F_n(t_1), \ldots, F_n(t_n)]^T\).

We are now ready to state the following theorem:

**Theorem 2.1** The algebraic systems (2.15) and (1.3) are equivalent.

**Proof**

It has been shown in [4] that \(Q_n = A_n^{-1} C_n\) and \(F_n = A_n^{-1} f\), where

\[(2.18) \quad (A_n^{-1})_{i,j} = n^{-1} (1-s_j^2)(t_i-s_j)^{-1}, \quad (A_n^{-1})_{i,i} = 1, \quad i,j = 1(1)n, \quad j = 1(1)n-1\]

After solving the algebraic system (2.16) or its equivalent (1.3) we obtain \(y_n^*(t_m)\), an approximation of the solution at the node points \(t_m\). For points different than \(t_m\), the Nyström interpolation formula (2.15) can be used directly. For points identical to the collocation points \(s_i\), the following modification of (2.15) should be used:
\[ 2.19 \quad y_n(s_i) = -\lambda n^{-1} \sum_{m=1}^{n} L_n(t_m, s_i) y_n(t_m) + F_n(s_i), \quad i=1(1)n-1, \]

where

\[ 2.20 \quad L_n(t_m, s_i) = -n^{-1} \sum_{k=1}^{n-1} (1-s_i^2)(s_k-s_i)^{-1}[K(s_k, t_m) - K(s_i, t_m)] + s_i k(s_i, t_m) \]

\[ = n^{-1} (1-s_i^2) \partial s K(s_i, t_m) \]

\[ 2.21 \quad F_n(s_i) = -n^{-1} \sum_{k=1}^{n-1} (1-s_i^2)(s_k-s_i)^{-1} [f(s_k) - f(s_i)] + s_i f(s_i) - n^{-1} (1-s_i^2) f'(s_i) + N. \]

3. **The Convergence of the Nyström Interpolant**

Let us consider $C[-1,1]$, the space of all continuous functions in $[-1,1]$, which is complete with the maximum norm:

\[ 3.1 \quad \|y\| = \max_{-1 \leq x \leq 1} |y(x)|. \]

We introduce the following linear operators on $C[-1,1]$

\[ 3.2 \quad L y = n^{-1} \int_{-1}^{1} (1-x^2)^{-1/2} L(x, t)y(x) dx \]

\[ 3.3 \quad L_n y = n^{-1} \sum_{m=1}^{n} L(t_m, t)y(t_m) \]

\[ 3.4 \quad L_n y^* = n^{-1} \sum_{m=1}^{n} L_n(t_m, t)y(t_m). \]

and define the norm of a linear operator $T$ by
5) \[ \|T\| = \sup_{\|y\| = 0} \|Ty\|/\|y\| = \epsilon_G \]

\[ y \in [-1, 1] \]

**Theorem 3.1** \[ \max_{-1 \leq x, t \leq 1} |L_n(x,t) - L(x,t)| \to 0, \text{ and } \|F_n - F\|_\infty \to 0 \text{ as } n \to \infty. \]

**Proof**

From (2.10), (2.14) we have that

\[ |L_n(x,t) - L(x,t)| = |r_n(K;x,t)|. \]

Equation (2.14) and the fact that \( g(t,x,s) \) is continuous implies that \( r_n(K;x,t) \) converges to zero pointwise (Szego [9], p. 350) in \([-1, 1] \times [-1, 1]\). To show that \( r_n(K;x,t) \) converges to zero uniformly, we notice that since

\[ A_1 = \int_{-1}^{1} \sqrt{1-s^2} \frac{1}{2} ds = \frac{\pi}{2} \quad \text{and} \quad C_1 = n^{-1} \pi \sum_{k=1}^{n-1} (1-s_k^2) = \frac{\pi}{2}, \]

3.7) \[ |r_n(K;x,t)| \leq (A_1 + C_1) \max_{-1 \leq x, t, s \leq 1} |g(x,t,s)| \]

3.8) \[ |r_n(K;x,t) - r_n(K;x^*, t^*)| \leq (A_1 + C_1) \max_{-1 \leq s \leq 1} |g(x,t,s) - g(x^*, t^*, s)|. \]

The continuity of \( g(x,t,s) \), and inequalities (3.7), (3.8) imply that the sequence \( (r_n(K;x,t)) \) is a uniformly bounded equicontinuous family of functions. Now by invoking the Arzela-Ascoli lemma, we can show that \( r_n(K;x,t) \) converges to zero uniformly (see [1], p. 92 for a similar argument).

The proof of the second part of the theorem is similar.
Theorem 3.2 \[ ||L_n - L^*|| \to 0 \text{ as } n \to \infty. \]

**Proof.** We have that,

\[
(3.9) \quad ||L_n - L^*|| \leq \max_{-1 \leq x, t \leq 1} |L(x,t) - L_n(x,t)|,
\]

and the theorem follows immediately.

**Theorem 3.3** For all \( y \in C[-1,1] \), \[ ||L_n y - Ly|| \to 0 \text{ as } n \to \infty. \]

**Proof.**

Since

\[
(3.10) \quad ||Ly|| \leq \max_{-1 \leq x, t \leq 1} |L(x,t)|
\]

\[
(3.11) \quad ||Ly(t) - Ly(t^*)|| \leq \max_{-1 \leq x, t \leq 1} |L(x,t) - L(x,t^*)|
\]

we have that the set \( \{Ly \mid ||y|| \leq 1\} \) is a bounded equicontinuous family of functions on \([-1,1]\), thus \( L \) is a compact operator from \( C[-1,1] \) to \( C[-1,1] \). Similarly

\[
(3.12) \quad ||L_n y|| \leq \max_{-1 \leq x, t \leq 1} |L(x,t)|
\]

\[
(3.13) \quad ||L_n y(t) - L_n y(t^*)|| \leq \max_{-1 \leq x, t \leq 1} |L(x,t) - L(x,t^*)|
\]

the sequence \( \{L_n y\} \) is a uniformly bounded equicontinuous set of functions which converges pointwise to \( Ly \) on \([-1,1]\) (Szegö [9], p. 350), thus \( \{L_n y\} \) must converge uniformly (see [1], p. 91).

**Remark 3.1** The sequence \( \{L_n\} \) is a collectively compact family of operators.

**Remark 3.2** In general \( ||L_n - L|| \to 0 \) as \( n \to \infty \), in fact

\[ ||L_n - L|| \leq \varepsilon \text{ and } ||L_n^2 - L^2|| \leq 2 \cdot ||L|| \text{ as } n \to \infty. \]
We rewrite equations (2.1), (2.4), (2.15) as follows:

\begin{align}
(3.14) \quad & (I + \lambda L)y = F \\
(3.15) \quad & (I + \lambda L_n)y_n = F \\
(3.16) \quad & (I + \lambda L_n^*)y_n^* = F_n
\end{align}

We will show that the Nyström interpolant

\begin{align}
(3.17) \quad & y_n^*(t) = -\lambda L_n^*y_n(t) + F_n(t)
\end{align}

converges to the solution \( y(t) \) of (3.14). To do so we need the following results:

**Theorem 3.4** If \( \lambda \) is not an eigenvalue of (3.14), then \( (I + \lambda L_n)^{-1} \) exists for all \( n \in \mathbb{N} \), and it is uniformly bounded by a constant \( B \), i.e. \( \| (I + \lambda L_n)^{-1} \| \leq B \).

**Proof** It follows directly from Theorem 3.3 (see [1], p. 98 and p. 105 for details).

**Corollary 3.1** Under the assumptions of Theorem 3.4 \( (I + \lambda L_n^*)^{-1} \) exist and it is uniformly bounded for all \( n \in \mathbb{N} \).

**Proof**

The identity

\begin{equation}
(3.18) \quad (I + \lambda L_n^*)^{-1} = (I + \lambda L_n)^{-1} + \left[ \frac{1}{\lambda} I - (I + \lambda L_n)^{-1}(L_n - L_n^*) \right]^{-1} (I + \lambda L_n)(L_n - L_n^*) (I + \lambda L_n)^{-1}
\end{equation}

and Theorem 3.4 shows that \( (I + \lambda L_n^*)^{-1} \) exists whenever

\begin{equation}
(3.19) \quad \left[ \frac{1}{\lambda} I - (I + \lambda L_n)^{-1}(L_n - L_n^*) \right]^{-1}
\end{equation}

exists, which is true since Theorem 3.2 implies that
3.20) \[ \| (I + \lambda L_n)^{-1}(L_n - L_n^*) \| \leq 8 \| L_n - L_n^* \| \| \lambda \|^{-1}, \text{ for } n \geq n_0. \]

The uniform boundedness is obvious.

**Remark 3.3** The last corollary and results given in [1], p. 105 imply that 
(I + \lambda C_n) exists. Moreover, the identity 
\[ (A_n + \lambda C_n)^{-1} = (I + \lambda A_n^{-1} C_n)^{-1} A_n^{-1} = (I + \lambda C_n)^{-1} A_n^{-1}, \]
and the existence of A_n^{-1} (shown in [4]), imply the existence of (A_n + \lambda C_n)^{-1} for sufficiently large n.

**Lemma 3.1** \[ \| y_n - y_n^* \| \to 0 \text{ as } n \to \infty. \]

**Proof**
The equation

\[ (I + \lambda L_n)(y_n - y_n^*) = \lambda (L_n^* - L_n) y_n^* + F - F_n \]

implies that

\[ \| y_n - y_n^* \| \leq \| (I + \lambda L_n)^{-1} \| \| \lambda \| \| L_n^* - L_n \| \| y_n^* \| + \| F - F_n \|. \]

The Lemma follows from Theorems 3.1, 3.2 and 3.4.

**Lemma 3.2** \[ \| y - y_n \| \to 0 \text{ as } n \to \infty. \]

**Proof**
The result follows from the inequality

\[ \| y - y_n \| \leq \| (I + \lambda L_n)^{-1} \| \| \lambda \| \| L_n - L_n^* \| \| y_n^* \| + \| F - F_n \|. \]

**Theorem 3.4** Assume that the kernel \( k(s, t) \in C^1([-1, 1] \times [-1, 1]) \) and \( f(s) \in C([-1, 1]). \)
If \( \lambda \) is not an eigenvalue of (1.1), then the Nyström interpolant \( y_n^*(t) \) defined
in (2.15) or (3.17) converges uniformly to the unique solution $y(t)$ of (1.1), (1.2).

Proof

Lemmas 3.1 and 3.2, and the inequality

\[(3.24) \quad || y_n^* ||_\infty \leq || y_n - y_n^* ||_\infty \leq || (I + \lambda L_n) ^{-1} || || (L_n - L_n) y_n^* || \leq \| F - F_n \| \]

proves the theorem.

Equation (3.24), gives us an upper bound for the error of $y_n^*(t)$. Moreover, we can see that the error bound depends on the quadrature error incurred when we approximate $Ly$ with $L_n y$, $L_n y$ with $L_n^* y$, $F$ with $F_n$. Since Gaussian-type quadratures have been used for the previously mentioned approximations, (3.24) implies that $|| y_n^* ||_\infty$ will converge to zero very fast, especially if the kernel and the input functions are smooth. We have successfully verified this observation on several numerical examples, three of which are described in the next section.

4. Numerical Examples and Error Bounds

(i) We first consider the singular integral equation of the form:

\[(4.1) \quad \pi^{-1} \int_1^{-1} (1-t^2)^{1/2} (t-s)^{-1} y(t) dt = f(s), \quad 1 \leq s \leq 1\]

\[(4.2) \quad \pi^{-1} \int_1^{-1} (1-t^2)^{1/2} y(t) dt = 0.\]

which arises in the stress analysis of a plane crack opened by the load distribution $f(s)$, in an isotropic medium \cite{4}. The solution $y(t)$ of (4.1), (4.2) represent the stress intensity factor at the tips of the crack. If we assume that $f(s) \in C \omega [-1, 1]$ then we can obtain an error bound for the Nyström interpolant by
using (2.8), (3.24), and the quadrature error formula [(2.12.6.6), p. 75, [3]].

Thus we have

\[ 4.3 \quad \|y - y_n^*\|_\infty \leq \|F - F_n\|_\infty \leq \max_{-1 \leq t \leq 1} \frac{3(2n-2)h(t,s)}{as(2n-2)} \left( \frac{(2n-2)2^{2(2n-1)}}{(2n-2)2(2n-1)} \right). \]

Using the Taylor series expansion of \( h(t,s) \), it can easily be shown that

\[ 4.4 \quad \max_{-1 \leq t \leq 1} \left| \frac{3(2n-2)h(t,s)}{as(2n-2)} \right| = \|f(2n-1)\|_\infty / (2n-1). \]

Finally, by combining (4.3), (4.4)

\[ 4.5 \quad \|y - y_n^*\|_\infty \leq \|f(2n-1)\|_\infty / [(2n-1)2(2n-1)]. \]

In Table 1 the numerical solution of equation (4.1), (4.2), with \( f(s) = \cos(s) \), at the point \( t = 1 \), is given. The "Actual Error" shown in the fourth column is the difference \( |y_n^*(1) - y(1)| \), while the "Maximum Error Bound" in the last column is computed by (4.5) with \( \|f(2n-1)\|_\infty = 1 \). For this example, and for others not reported in this paper, the error bound given in (4.5) agrees extremely well with the actual computational error. Moreover, by comparing the second and third column we see that the Nyström interpolant converges much faster than Krenk's interpolation formula.

The error bound given in (4.5), shows that Nyström's interpolation formula will yield excellent results, especially for smooth input functions.

Note: The exact solution \( y(1) \) is obtained to within any degree of accuracy by inverting (4.1), (4.2) and applying the Gauss-Chebyshev quadrature, i.e.

\[ 4.6 \quad y(1) = \pi^{-1} \int_0^1 (1-s^2)^{-1/2} (1+s) \cos(s) ds = -1 \sum_{m=1}^{n} \frac{(1+t_m) \cos(t_m)}{m} = 0.7651976865579665 \]

for all \( n \geq 10 \).
Table 1

\( y(1) \)

<table>
<thead>
<tr>
<th>n</th>
<th>Krenk's interpolation [7]</th>
<th>Nyström interpolation (2.15)</th>
<th>Actual Error of (2.15)</th>
<th>Maximum Error Bound (4.5)</th>
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<td>1.10^{-18}</td>
<td>2.310^{-17}</td>
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To provide a further comparison of the different type of interpolation formulas and methods, we consider two integral equations arising in the solution of Elasticity problems. The equation:

\( 1 \) \( \pi^{-1} \int_{-1}^{1} \frac{1}{(1-t^{2})} \left( \frac{1}{2} y(t) \right) dt = \lambda \int_{-1}^{1} \frac{1}{(1-t^{2})} y(t) dt = 1, -1 < s < 1 \)  

\( 2 \) \( \pi^{-1} \int_{-1}^{1} \frac{1}{(1-t^{2})} y(t) = 0, \)  

arises in the solution of a cover plate bonded to an elastic half-space [2], and the equation:

\( 3 \) \( \pi^{-1} \int_{-1}^{1} \frac{1}{(1-t^{2})} \left( \frac{1}{2} y(t) \right) dt + \pi^{-1} \int_{-1}^{1} \frac{1}{(1-t^{2})} \left( \frac{1}{2} y(t) \right) dt = 1, -1 < s < 1 \)  

\( 4 \) \( \pi^{-1} \int_{-1}^{1} \frac{1}{(1-t^{2})} y(t) = 0, \)  

arises in the solution of a cruciform crack [10].
From Tables 2 and 3 we can see that the Nyström interpolant converges at least as fast or even faster than all methods considered, even though the kernels $K(s,t)$, in both cases are not continuous. The reason that the Gauss-Chebyshev method converges here is that the error in the Gaussian quadrature approximating the kernel part of (1.1) depends on the smoothness of $K(s,t)y(t)$, rather than the smoothness of the kernel $K(s,t)$ itself. Although we have considered the Nyström interpolation formula only for the point $t = 1$, the formula can be applied just as easily for all points in $[-1, 1]$, except for the nodes $t = t_i$.

**Table 2**

Equation (4.3), (4.4). The strength of Stress Singularity $y(1)$

<table>
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<tr>
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<td>0.4108</td>
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</tr>
</tbody>
</table>

**Table 3**

Equation (4.5), (4.6). The stress intensity factor $y(1)$

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<td>60</td>
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<td>0.86355</td>
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Note. All calculations have been performed in DEC-20 FORTRAN with double precision arithmetic.
References


