198:205
Discrete Structures I
Professor McCarty

Solutions to Sample Midterm Exam

Question 1:

Here is the truth table for part (a):

$x \in A$	$x \in B$	$x \in A \oplus x \in B$
T	T	F
T	F	T
F	T	T
F	F	F

Here is the truth table for part (b), drawn as the horizontal continuation of the truth table for part (a):

$x \in A \vee x \in B$	$x \in A \wedge x \in B$	$\neg(x \in A \wedge x \in B)$	$(x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)$
T	T	F	F
T	F	T	T
T	F	T	T
F	F	T	F

Since the last two columns in each table are the same, the two expressions are logically equivalent.

For each subpart of part (c), we start with the logical equivalence from part (b), derive additional logical equivalences as needed, and then translate the results into a statement about sets using set builder notation.

(i) In this case, we use the logical equivalence in part (b) directly:

$$\begin{aligned}
A \oplus B &= \{x \mid x \in A \oplus x \in B\} \\
&= \{x \mid (x \in A \vee x \in B) \;\wedge\; \neg(x \in A \wedge x \in B)\}, \text{ by part (b),} \\
&= \{x \mid (x \in A \cup B) \wedge \neg(x \in A \cap B)\}, \text{ by the definition of } \cup \text{ and } \cap, \\
&= (A \cup B) - (A \cap B), \text{ by the definition of set difference.}
\end{aligned}$$

(ii) In this case, starting with part (b), we first derive a new logical equivalence:

$$\begin{aligned}
&x \in A \oplus x \in B \\
&\qquad \equiv (x \in A \vee x \in B) \wedge \neg (x \in A \wedge x \in B), \quad \text{by part (b)}, \\
&\qquad \equiv (x \in A \vee x \in B) \wedge (\neg x \in A \vee \neg x \in B), \quad \text{by De Morgan's law}, \\
&\qquad \equiv ((x \in A \vee x \in B) \wedge \neg x \in A) \vee ((x \in A \vee x \in B) \wedge \neg x \in B), \\
&\qquad \quad \text{by the distributive law}, \\
&\qquad \equiv (\neg x \in A \wedge (x \in A \vee x \in B)) \vee (\neg x \in B \wedge (x \in A \vee x \in B)), \\
&\qquad \quad \text{by the commutative law}, \\
&\qquad \equiv ((\neg x \in A \wedge x \in A) \vee (\neg x \in A \wedge x \in B)) \vee (\neg x \in B \wedge (x \in A \vee x \in B)), \\
&\qquad \quad \text{by the distributive law}, \\
&\qquad \equiv (\mathbf{F} \vee (\neg x \in A \wedge x \in B)) \vee (\neg x \in B \wedge (x \in A \vee x \in B)), \\
&\qquad \quad \text{by the law of contradiction}, \\
&\qquad \equiv (\neg x \in A \wedge x \in B) \vee (\neg x \in B \wedge (x \in A \vee x \in B)), \\
&\qquad \quad \text{by the commutative and identity laws}, \\
&\qquad \equiv (\neg x \in A \wedge x \in B) \vee ((\neg x \in B \wedge x \in A) \vee (\neg x \in B \wedge x \in B)), \\
&\qquad \quad \text{by the distributive law}, \\
&\qquad \equiv (\neg x \in A \wedge x \in B) \vee ((\neg x \in B \wedge x \in A) \vee \mathbf{F}), \\
&\qquad \quad \text{by the law of contradiction}, \\
&\qquad \equiv (\neg x \in A \wedge x \in B) \vee (\neg x \in B \wedge x \in A), \quad \text{by the identity law}, \\
&\qquad \equiv (x \in A \wedge \neg x \in B) \vee (x \in B \wedge \neg x \in A), \quad \text{by the commutative laws.}
\end{aligned}$$

We then use this equivalence inside our set builder notation:

$$\begin{aligned}
A \oplus B &= \{x \mid x \in A \oplus x \in B\} \\
&= \{x \mid (x \in A \wedge \neg x \in B) \vee (x \in B \wedge \neg x \in A)\}, \\
&\quad \text{by the equivalence derived above}, \\
&= \{x \mid (x \in A - B) \vee (x \in B - A), \quad \text{by the definition of set difference}, \\
&= (A - B) \cup (B - A), \quad \text{by the definition of } \cup.
\end{aligned}$$

(iii) In this case, we first derive:

$$x \in A \ \oplus \ x \in B$$

$$\begin{aligned}
&\equiv (x\in A \vee x \in B) \wedge \neg\, (x\in A \wedge x\in B), \text{ by part (b)},\\
&\equiv (\neg\neg\, x\in A \vee \neg\neg\, x \in B) \wedge \neg\, (x\in A \wedge x\in B), \text{ by double negation},\\
&\equiv \neg\, (\neg\, x\in A \wedge \neg\, x \in B) \wedge (\neg\, x\in A \vee \neg\, x\in B), \text{ by De Morgan's law},\\
&\equiv \neg\, (x\in \overline{A} \wedge x \in \overline{B}) \wedge (x\in \overline{A} \vee x\in \overline{B}), \text{ by definition of complement},\\
&\equiv (x\in \overline{A} \vee x\in \overline{B}) \wedge \neg\, (x\in \overline{A} \wedge x \in \overline{B}), \text{ by the commutative law},\\
&\equiv x\in \overline{A} \;\oplus\; x\in \overline{B}, \text{ by part (b)}.
\end{aligned}$$

It is then obvious how to use this equivalence inside our set builder notation:

$$\begin{aligned}
A \oplus B &= \{x \mid x\in A \oplus x\in B\}\\
&= \{x \mid x\in \overline{A} \oplus x\in \overline{B}\}\\
&= \overline{A} \oplus \overline{B}.
\end{aligned}$$

Question 2:

1. We will use logical equivalences to transform the left-hand side (reading down) and the right-hand side (reading up), and then use an informal but rigorous argument to bridge the gap. The left-hand side is:

   $$\exists x \exists y (P(x) \rightarrow Q(y)) \qquad \equiv$$

   $$\exists x \exists y (\neg P(x) \vee Q(y)) \tag{1}$$

   The right-hand side is:

   $$\exists x \neg P(x) \vee \exists y Q(y) \qquad \equiv \tag{2}$$

   $$\neg \forall x P(x) \vee \exists y Q(y) \qquad \equiv$$

   $$\forall x P(x) \rightarrow \exists y Q(y)$$

   To complete the proof, we need to show that (1) is logically equivalent to (2).

   Assume (1) is true. Then there is some x_0 such that $\exists y(\neg P(x_0) \vee Q(y))$, and thus some y_0 such that $\neg P(x_0) \vee Q(y_0)$. We can reason by cases with this disjunction.

If $\neg P(x_0)$ is true, then $\exists x \neg P(x)$ is true, and if $Q(y_0)$ is true, then $\exists y Q(y)$ is true. In either case, though, we can derive $\exists x \neg P(x) \vee \exists y Q(y)$, which is (2).

Conversely, assume (2) is true and reason by cases with this disjunction. If $\exists x \neg P(x)$ is true, then there is some x_0 such that $\neg P(x_0)$ is true. If $\exists y Q(y)$ is true, then there is some y_0 such that $Q(y_0)$ is true. In either case, though, $\neg P(x_0) \vee Q(y_0)$ is true. Thus, restoring the existential quantifiers in order, we derive first $\exists y(\neg P(x_0) \vee Q(y))$ and then $\exists x \exists y(\neg P(x) \vee Q(y))$, which is (1).

2. This is also a logical equivalence, which can be shown directly from the implicational form.

   First, assume $\forall x \forall y(P(x) \to Q(y))$. We want to show that the implication on the right-hand side of the equivalence is true, so we assume that the antecedent of this implication, $\exists x P(x)$, is true. From this assumption, it follows that there is some x_0 such that $P(x_0)$ is true, and from our main assumption we can now derive $\forall y(P(x_0) \to Q(y))$. We would like to derive $\forall y Q(y)$, however, so we choose an arbitrary y_0 and try to derive $Q(y_0)$. From our prior derivation of $\forall y(P(x_0) \to Q(y))$, we know that $P(x_0) \to Q(y_0)$ is true, and since $P(x_0)$ is true, it follows that $Q(y_0)$ is true. Finally, since y_0 was arbitrary, we have succeeded in showing that $\forall y Q(y)$ must be true as well. *QED*

   Conversely, assume $\exists x P(x) \to \forall y Q(y)$. We want to show that the universally quantified implication on the left-hand side of the equivalence is true. To do this, we choose an arbitrary x_0 and try to show that $\forall y(P(x_0) \to Q(y))$ is true, and to do this, in turn, we choose an arbitrary y_0 and try to show that $P(x_0) \to Q(y_0)$ is true. We are now trying to prove an implication, so we assume that the antecedent, $P(x_0)$, is true. From this assumption, it follows that $\exists x P(x)$ is true, and then from our main assumption it follows that $\forall y Q(y)$ is true. Therefore, for our chosen y_0, $Q(y_0)$ is true. We have now shown that $P(x_0) \to Q(y_0)$ is true, and since both x_0 and y_0 were arbitrary, we have succeeded in showing that $\forall x \forall y(P(x) \to Q(y))$ must be true as well. *QED*

Question 3:

1. First, assume f is one-to-one. Let

$$f(A) = \{b \mid f(a) = b \text{ and } a \in A\} \subseteq B$$

be the range of f in B. Since f is one-to-one, the set $\{x \mid f(x) = b\}$ is a singleton set for every $b \in f(A)$. We can thus choose an arbitrary element $a_0 \in A$, and define $g : B \mapsto A$ as follows:

$$\begin{aligned} g(b) &= \text{the unique element } a \in \{x \mid f(x) = b\}, \text{ if } b \in f(A), \\ &= a_0, \text{ if } b \notin f(A). \end{aligned}$$

Clearly, g is a well-defined function, i.e., it is single-valued on B, and it satisfies the property $g \circ f = \iota_A$, i.e., it is a left inverse of f.

Conversely, assume that there exists a function $g : B \mapsto A$ such that $g \circ f = \iota_A$. We will show that f is one-to-one. Suppose $f(x) = f(y)$ for some elements x and y in A. Then:

$$\begin{aligned} g(f(x)) &= g(f(y)) \\ (g \circ f)(x) &= (g \circ f)(y) \\ \iota_A(x) &= \iota_A(y) \\ x &= y \end{aligned}$$

Thus, f is one-to-one.

2. No, a left inverse is not unique. We can see this in any set A with at least two distinct elements a_1 and a_2. Define functions g_1 and g_2 which are exactly like g, as defined above, except that the element $a_0 \in A$ is replaced by the elements a_1 and a_2, respectively. Then both g_1 and g_2 will be left inverses of f, but they will not be equal, since they will produce different values for every $b \notin f(A)$.