198:205
Discrete Structures I
Professor McCarty

Sample Midterm Exam

Advice: Read each problem, think about it, read it again, and think about it again *before* you start writing. If you have a question, ask! Don't get bogged down on a particular problem. If you're stuck, go on to the next problem, and come back again later.

Question 1:

The *symmetric difference* of two sets A and B, denoted $A \oplus B$, is the set containing those elements that are either in A or in B, but are not in both A and B. Formally, the symmetric difference would be defined using set builder notation together with exclusive disjunction as follows:

$$A \oplus B \;=\; \{x \mid x \in A \oplus x \in B\}$$

You should use this formal definition in answering the following questions:

(a) Write out the truth table for $x \in A \oplus x \in B$.

(b) Use truth tables to show that

$$x \in A \oplus x \in B \;\equiv\; (x \in A \vee x \in B) \;\wedge\; \neg(x \in A \wedge x \in B)$$

(c) Use the logical equivalence in (b), plus any other logical equivalences that we established in class or in the text, to show that:

(i) $A \oplus B \;=\; (A \cup B) - (A \cap B)$.

(ii) $A \oplus B \;=\; (A - B) \cup (B - A)$.

(iii) $A \oplus B \;=\; \overline{A} \oplus \overline{B}$.

Question 2:

Which of the following pairs of formulae are logically equivalent? In each case, consider, first, the implication from left to right. If it is true, write out a proof. If it is false, give a countermodel. Then do the same for the implication from right to left. In your proofs, you may use any equivalences and implications that we established in class, but be sure to justify each step.

1. $\exists x \exists y (P(x) \rightarrow Q(y)) \quad ?\equiv? \quad \forall x P(x) \rightarrow \exists y Q(y)$
2. $\forall x \forall y (P(x) \rightarrow Q(y)) \quad ?\equiv? \quad \exists x P(x) \rightarrow \forall y Q(y)$

Question 3:

Let f be a function from a nonempty set A to a set B, $f : A \mapsto B$. If there exists a function $g : B \mapsto A$ such that $g \circ f = \iota_A$, then g is called a *left inverse* of f.

1. Prove that f has a left inverse if and only if f is one-to-one.
2. Is a left inverse unique? If your answer is "Yes", give a proof. If your answer is "No", give a counter example.