In How Many Steps the k Peg Version of the Towers of Hanoi Game Can Be Solved?

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Abstract. In this we paper we consider the version of the classical Towers of Hanoi games where the game-board contains more than three pegs. For k pegs we give a $2^{C_k n^{1/(k-2)}}$ lower bound on the number of steps necessary for transferring n disks from one peg to another. Apart from the value of the constants C_k this bound is tight.

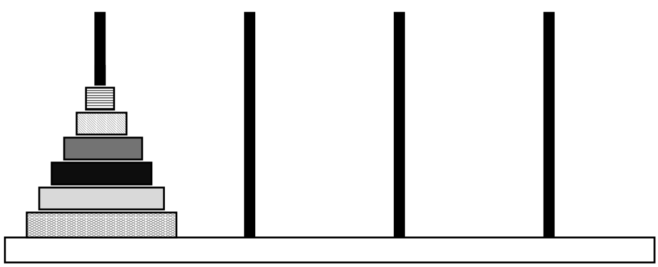

Fig. 1. The board of the Towers of Hanoi game with four pegs

1 Introduction

"In an ancient city, so the legend goes, monks in a temple had to move a pile of 64 sacred disks from one location to another. The disks were fragile; only one could be carried at a time. A disk could not be placed on top of a smaller, less valuable disk. In addition, there was only one other location in the temple (besides the original and destination locations) sacred enough for a pile of disks to be placed there.

Using the intermediate location, the monks began to move disks back and forth from the original pile to the pile at the new location, always keeping the piles in order (largest on the bottom, smallest on the top). According to the legend, before the monks could make the final move to complete the new pile in the new location, the temple would turn to dust and the world would end." [12]

C. Meinel and S. Tison (Eds.): STACS'99, LNCS 1563, pp. 356–361, 1999.

There is a single player game, called "The Towers of Hanoi," originating from the above legend with the following rules:

The game board has three pegs and n disks, which are originally arranged on the first peg, largest at the bottom and each disk sitting on a larger disk. The goal of the game is to transfer the n disks to another peg while observing the following conditions:

1. In each step the topmost disk on a peg is removed and placed on the top of the disks on another peg.
2. A disk cannot be placed on a disk smaller than itself.

The game has become one of the most popular examples for recursive algorithms (see e.g. [1], [6]). The unique shortest solution requires $2^n - 1$ steps. Since there is not much mathematical mystery left about the original game, its lovers developed various versions of it [3], [4], [9], [2]. In this article we consider a version where the number of pegs is some fixed $k > 3$ ([3], [4]). Define $h = k - 2$, and s as the unique integer for which $\binom{h+s-1}{h} < n \leq \binom{h+s}{h}$. In [3] and [7] different algorithms are presented that require exactly

$$a_k(n) = 2^s \left(n - \binom{h+s-1}{h} \right) + \sum_{t=0}^{s-1} 2^t \binom{h+t-1}{h} \tag{1}$$

disk moves when transferring the pile of disks from the first peg to the second peg. To understand the above formula a little better observe that $a_k(n) - a_k(n-1) = 2^s$, where s is defined as above. This amount is perceived as the increase in the number of steps caused by the presence of the n^{th} disk. (The formula appearing in [3] is equivalent to (1)) The order of magnitude of $a_k(n)$ is $2^{B_k n^{1/(k-2)}}$ where $B_k = (1 \pm o(1))(k-2)!^{1/(k-2)}$.

Our main result (Corollary 2) is a $2^{(1 \pm o(1))C_k n^{1/(k-2)}}$ lower bound for the number of necessary moves to solve the k pegs version of the Towers of Hanoi game, which is optimal up to a constant factor in the exponent for fixed k.

Table 1. The values of $a_k(n)$ for $k \leq 5$ and $n \leq 10$.

$k \backslash n$	1	2	3	4	5	6	7	8	9	10
3	1	3	7	15	31	63	127	255	511	123
4	1	3	5	9	13	17	25	33	41	49
5	1	3	5	7	11	15	19	23	27	31

2 Motivation

The question about the optimal number of steps for the k peg variant of the Towers of Hanoi game was raised in the American Mathematical Monthly in

1939 (Problem 3918). The two solutions that it received in 1941 ([5], [10]) claim that the exact bound is $a_k(n)$, but the proofs are valid only for a very restricted class of algorithms. No unconditional lower bound has existed so far even though a few computer scientists have implemented algorithms for the problem (see e.g. [8], [13] for a nice web version). Since only a minimal background is required to understand the topic, the design of such algorithms is and ideal project for interested students.

The lower bound proof we present may also serve as a toy example for more involved complexity theoretic lower bound proofs, and its structure may be worthwhile to study for its own sake.

3 The Lower Bound Proof

Our lower bound will hold in a more general setup, namely when our only requirement is that between the initial and the final configuration each disk moves at least once. This generalization will allow us to use an induction. Getting the precise bound on the original problem might require a different kind of argument.

Definition 1. *For a game board with k pegs an arrangement of n disks is called a* configuration *if it obeys the "smallest disk on top of the larger" rule. For a configuration D as above let $g(D)$ be the minimal number of steps required to get every disk moved at least once, where all moves are taken according to rules 1. and 2. of the introduction. Let us define $g(n,k) = \min_D g(D)$ where D runs though over all possible configurations of n rings on a game board with k pegs.*

Remark 1. $g(D)$ is finite for every configuration D.

Proof. Since $g(D_0)$ is known to be finite, where D_0 is the the configuration where all disks pile up on the first peg, it is enough to show that this configuration can be reached from every other configuration. We can show this by using an induction on n. □

Theorem 1. *For $k \geq 3$*

$$g(n,k) \geq 2^{(1 \pm o(1))C_k n^{1/(k-2)}}.$$

Here the constants C_k depend on k in the following way:

$$C_k = \frac{1}{2}\left(\frac{12}{k(k-1)}\right)^{1/(k-2)}.$$

Proof. We proceed by induction on k.

Remark 2. We shall make a little effort to get the optimal lower bound for $g(n,3)$, even though it would be enough for us to show that $g(n,3) = \Omega(2^n)$. We shall show that $g(n,3) \geq 2^{n-2} + 1$. From the configuration where the largest and the second largest disks are around the first peg, and all other disks are around the second peg we can see that this bound is sharp.

The case $k = 3$: We can suppose that $n \geq 2$. Consider an arbitrary initial arrangement of n disks. Let $\mathcal{S}$ be a sequence of the steps in which every disk moves at least once. Let j denote the first step that moves the largest disk. Let $\mathcal{S}_1$ be the sequence of steps that proceed j, and let $\mathcal{S}_2$ be the sequence of all steps following j. In the configurations before and after step j the disks other than the largest disk are piled up in a single tower. On the other hand, by our assumption, in either $\mathcal{S}_1$ or in $\mathcal{S}_2$ the second largest disk (which is at the bottom of the pile before and after step j) should move at least once. By symmetry we can assume that this happens during the moves of $\mathcal{S}_2$.

It is easy to see that then $\mathcal{S}_2$ must contain a solution to the three peg Towers of Hanoi problem on the $n-2$ smallest disks, so $\mathcal{S}_2$ must be at least $2^{n-2}-1+1$ long. Here $2^{n-2}-1$ comes from the lower bound for the classical game of Hanoi (see e.g. [6]) to which we can add one, because the second largest disk must also move. $\mathcal{S}$ contains at least one more step than $\mathcal{S}_2$ (namely step j), so we obtain: $g(n,3) \geq 2^{n-2}+1$.

The case $k \geq 4$: First we prove a lemma which serves as the main lemma in our argument:

Lemma 1. *Suppose $k \geq 4$ and $0 < m < n/2k$. Then:*

$$g(n,k) \geq 2\min(g(n-2km,k), g(m,k-1)).$$

Proof. We say that a sequence $\mathcal{S}$ of steps *moves* a set H of disks if for every $h \in H$ there exists a step in $\mathcal{S}$ made by h. Let us call Z (small disks) the set of the smallest $n-2km$ disks.

Consider an arbitrary configuration $\mathbf{D}$ of the disks around k pegs. There is a peg around which we have at least $2m$ disks from the largest $2km$ disks. Let us call X (extra large disks) the set of the largest m disks in this peg. Let us call L (large disks) the set of the next largest m disks in this peg . Note that X and L depend on $\mathbf{D}$ while Z does not. Clearly Z, L and X are disjoint, $\mid X \mid = \mid L \mid = m$, $\mid Z \mid = n-2km$. Moreover every disk in X is larger than any disk in L and they are all larger than any disk in Z. Consider a sequence of steps that moves all the disks starting from the initial configuration $\mathbf{D}$. Define $\mathcal{S}_1$ as the initial sequence of steps up to (but excluding) the first step by the topmost (i.e. the smallest) disk of X, and let $\mathcal{S}_2$ be the sequence of all remaining steps. Obviously $\mathcal{S}_1$ moves L and $\mathcal{S}_2$ moves X. Moreover if $\mathcal{S}_1$ does not move Z then the peg on which an idle member of Z is sitting is completely useless for the disks in L since they are all larger than any of the disks in Z.

This allows us in this case to estimate the number of steps made by the elements of L by $g(m,k-1)$. The same argument shows that $\mathcal{S}_2$ either moves Z or only $k-1$ pegs were used when making the steps with the disks in X. Since according to the above argument both $\mathcal{S}_1$ and $\mathcal{S}_2$ contain at least $\min(g(n-2km,k), g(m,k-1))$ steps the proof of the lemma follows. □

Let us denote $\log_2 g(n,k)$ by $\phi(n,k)$.

Corollary 1. *For $k \geq 4$ and $0 < m < n/2k$*

$$\phi(n,k) \geq 1 + \min(\phi(n-2km,k), \phi(m,k-1))$$

holds.

Lemma 2. *$\phi(n,k)$ is a monotone increasing function of n for any fixed $k \geq 3$.*

Lemma 2 follows from the fact that extra disks just make the task harder.

Lemma 3. *Suppose $k \geq 4$ and $\phi(n_i, k-1) \geq i$ for $i = 1,..,s$. Then*

$$\phi(2k\sum_{i=1}^{s} n_i, k) \geq s.$$

Proof. We proceed by induction on s. The $s = 1$ case is straightforward. For $s \geq 2$:

$$\phi(2k\sum_{i=1}^{s} n_i, k) = \phi(2kn_s + 2k\sum_{i=1}^{s-1} n_i, k) \geq$$

$$1 + \min(\phi(2k\sum_{i=1}^{s-1} n_i, k), \phi(n_s, k-1)) \geq s.$$

The first inequality comes from the corollary of Lemma 1, the second comes from the induction hypothesis on $s-1$ and the assumption of the lemma on $\phi(n_s, k-1)$. □

Lemma 4. *Suppose the Theorem holds for $k-1$. Define n_i as the smallest element of the set $\{n \mid \phi(n,k-1) \geq i\}$. Then:*

$$\sum_{i=1}^{s} n_i \leq (1 \pm o(1))\frac{s^{k-2}}{(k-2)C_{k-1}^{k-3}}.$$

Proof. Since according to our assumption the theorem holds for $k-1$ we have the $(1 \pm o(1))C_{k-1}n^{1/k-3}$ lower bound on $\phi(n,k-1)$. Combining this with the monotonicity of $\phi(n,k-1)$ in n we get the asymptotic upper bound on the integral of the inverse required by the lemma. □

Now we are ready to prove the Theorem 1 for $k \geq 4$ assuming that it is true for $k-1$. Let us denote $\sum_{i=1}^{s} n_i$ by N_s. From Lemma 4 we have $N_s \leq (1 \pm o(1))\frac{2ks^{k-2}}{(k-2)C_{k-1}^{k-3}}$. On the other hand Lemma 3 says that $\phi(2kN_s, k) \geq s$. for every s. These two inequalities provide us with an asymptotic upper bound on the inverse of $\phi(n,k)$, which can be easily turned into the following asymptotic lower bound on $\phi(n,k)$ using the monotonicity in n:

$$\phi(k,n) \geq (1 \pm o(1)){C_{k-1}}^{\frac{k-3}{k-2}}(\frac{k-2}{2k})^{1/(k-2)}n^{k-2}.$$

Calculation shows that $C_k = {C_{k-1}}^{\frac{k-3}{k-2}}(\frac{k-2}{2k})^{1/(k-2)}$. □

Corollary 2. *The k peg version of the Towers of Hanoi problem requires at least* $2^{(1-o(1))C_k n^{1/(k-2)}}$ *steps.*

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