Geometric representation of cubic graphs with four directions

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Abstract

We show that every connected cubic graph can be can be drawn in the plane with straight line edges using only four distinct slopes.

1 Introduction

A drawing of a graph is said to be a *straight-line drawing* if the vertices of G are represented by distinct points in the plane and every edge is represented by a straight-line segment connecting the corresponding pair of vertices and not passing through any other vertex of G. Wade and Chu [WC94] defined the *slope number* $sl(G)$ of a graph G to be the smallest number of distinct slopes used in a straight-line drawing of the graph. We may note that a vertex of degree d requires atleast $\lceil \frac{d}{2} \rceil$ slopes and hence, the bound on the slope number would depend on the maximum degree of the graph. Dujmović et al. [DSW04] asked if the slope number of a graph with bounded maximum degree could be arbitrarily large. Pach and Pálvölgyi [PP06] and Barát, Matoušek, Wood [BMW05] (independently) showed with a counting argument that the number of degree-5 graphs (graphs with maximum degree 5) exceeds the number of graphs drawn with a fixed number of slopes, thereby proving that a finite number of slopes are insufficient to draw all degree-5 graphs.

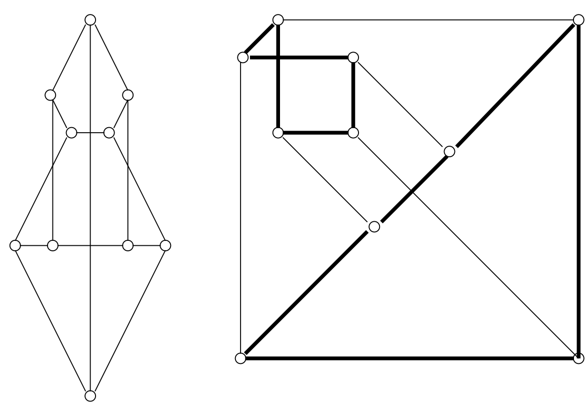

Figure 1: Drawings of the peterson graph with four slopes

In [KPPT] it was shown that cubic graphs could be drawn with five slopes. This involved showing (inductively) that any subcubic graph (a degree-3 graph with at least one vertex of degree two or less) can be drawn with four slopes. Therefore, a cubic graph requires one additional slope. We show in this paper that four slopes suffice.

Theorem 1.1 *Every connected cubic graph has a straight line drawing with only four slopes.*

It was shown by Max Engelstein [E05] that cubic graphs with a Hamiltonian cycle can be drawn with four slopes. Slope numbers of other subclasses of graphs like outerplanar graphs, 2-trees, 3-trees, 3-connected planar graphs have been discussed by Dujmović et al. in [DSW07]. In particular, they show that every cubic 3-connected planar graph has a plane drawing with three slopes.

The *geometric thickness* of a graph is the smallest number of planar graphs a straight-line drawing of a graph can be decomposed into. For a straight-line drawing of a graph, it is easy to see that the subgraph obtained by picking all the edges with the same slope is planar. Hence, the slope number provides a natural upper bound for the geometric thickness of a graph. Barát, Matoušek, Wood in [BMW05] also show that the geometric thickness of degree-9 graphs is unbounded. Duncan et al. [DEK04] show that the geometric thickness of degree-4 graphs is 2.

Section 2 of this paper deals with an outline of the proof of Theorem 1.1. Section 3 is dedicated to proofs for some claims made in the second section. Open problems are discussed in the final section.

2 Outline of the proof

2.1 Assumptions

We will assume in the rest of the paper that the graph is bridgeless and triangle-free. We will use the following theorem to see why these assumptions are valid.

Theorem 2.1 ([KPPT]) *Let G be a connected graph that is not a cycle and whose every vertex has degree atmost three. Suppose that G has atleast one vertex of degree less than three and denote by $v_1, ..., v_m$ the vertices of degree atmost two ($m \geq 1$).*

Then, for any sequence $x_1, ..., x_m$ of real numbers, linearly independent over the rationals, G has a straight-line drawing with the following properties:

(1) Vertex v_i is mapped into a point with x-coordinate $x(v_i) = x_i \quad (1 \leq i \leq m)$

(2) The slope of every edge is $0, \pi/2, \pi/4$, or $-\pi/4$

(3) No vertex is to the North of any vertex of degree two.

(4) No vertex is to the North or to the Northwest of any vertex of degree one.

We would use this theorem to patch together different components of a cubic graph obtained after removal of some edges. For this we would want to note that we could rotate the components by any multiple of $pi/4$ and still have a graph with the same slopes as before.

Claim 2.2 *A cubic graph with a bridge or a minimal two-edge disconnecting set can be drawn with four slopes.*

Proof. Both components obtained by removing the bridge can be drawn with four slopes using Theorem 2.1. Both have the north direction free for the vertex of degree two. To put these together, rotate the second one by $\pi/2$ and place the degree two vertices above each other. Move the components far enough so that none of the other vertices or edges overlap.

For a two-edge disconnecting set, we may note that these edges must be vertex-disjoint or the graph would contain a bridge. Then, the same procedure as above can be used, now keeping the distance between the two vertices of degree two the same in both components. This method cannot be extended to a minimal disconnecting set with more edges as then one of the components might be a cycle and then the above theorem cannot be invoked. □

Claim 2.3 *A cubic graph with a cutvertex or a two-vertex disconnecting set can be drawn with* 4 *slopes.*

Proof. If the graph has a cutvertex, then it has a bridge. If it has a two-vertex disconnecting set, then it has a two-edge disconnecting set. In both cases we can then invoke Claim 2.2 to draw the graph with four slopes. □

Claim 2.4 *Any cubic graph with a triangle can be drawn with* 4 *slopes.*

Proof. First we note that by the above claims, we may assume that we only consider cubic graphs in which trianges are connected to the rest of the graph by vertex disjoint edges. If not, then the graph is either K_4 or has a two-vertex disconnecting set and in either case, we can draw it with four slopes.

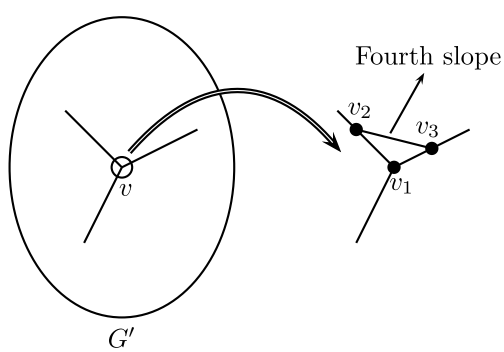

Figure 2: Adding the triangle to the drawing of G' with four slopes

Now we prove the claim by contradiction. Suppose there exist cubic graphs with triangles that cannot be drawn with four slopes. Of all such graphs consider the one with minimum number of vertices. The graph obtained by contracting the edges of the triangle $\{v_1, v_2, v_3\}$ is also cubic, has fewer vertices, and therefore, can be drawn with four slopes. Call the vertex formed by contracting the edges of the triangle as v. Since there is one slope that is not used up by the edges incident on v, draw a segment with this slope in an very small neighbourhood of v as shown in the figure. This contradicts the existence of a minimal counterexample and hence all graphs with triangles can be drawn with four slopes.

□

Remark 2.5 *It must be noted that this also gives an algorithm for drawing cubic graphs with triangles, namely, we contract triangles until we get a graph that can be drawn with either the Claims 2.2,2.3,2.4 or Theorem 2.1 or with our drawing strategy for triangle-free bridgeless graphs. Then we can backtrack with placing a series of edges which give us back all the contracted triangles.*

Remark 2.6 *A consequence of the above discussion is that any cubic graph that cannot be drawn with the* **standard** *four slopes (N,E,NE,NW) must be three vertex and edge connected.*

2.2 Drawing strategy

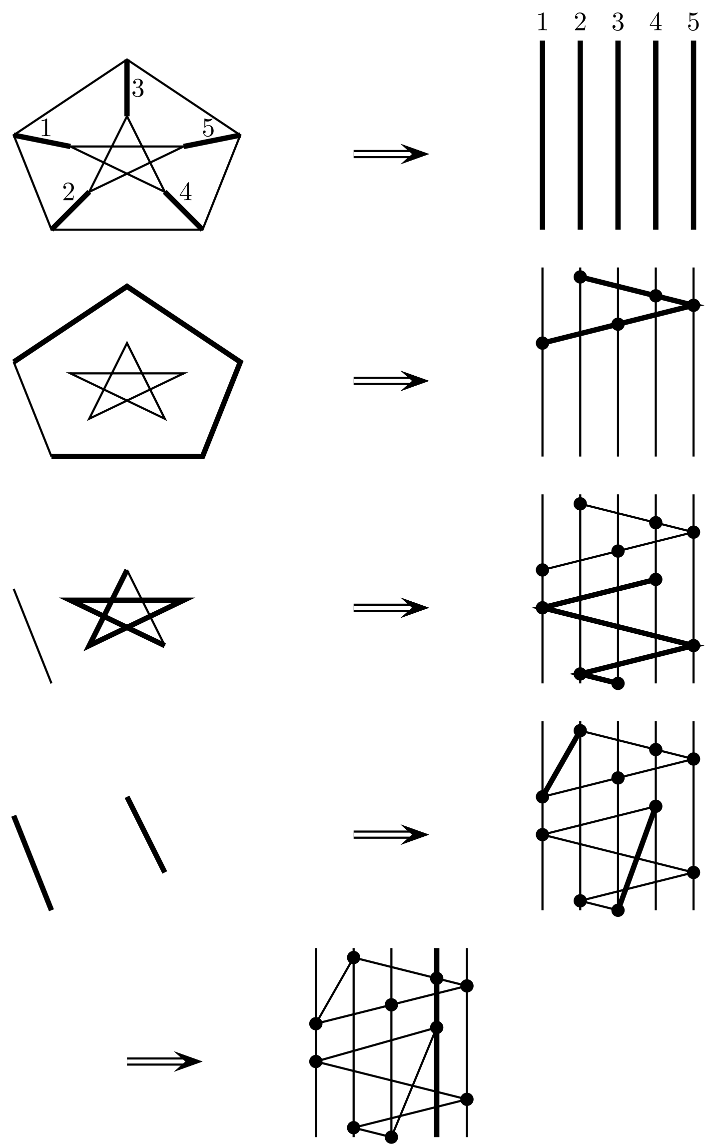

Figure 3: Process of drawing the cycles

Because of the above claims, we would now only focus on graphs that are bridgeless and triangle-free. Since the graph is bridgeless, Peterson's theorem implies that it has a matching.

We fix the slope of all the edges in the matching to be $\pi/2$ so that they all lie on (distinct) vertical lines (Figure 3). If this matching is removed, then the graph consists of disjoint cycles. Next we isolate one special edge from each cycle. Our method of drawing the graph with four slopes then is as follows: For each cycle, remove the selected edge and draw the remaining path by going between corresponding vertical lines of the cycle alternating with slopes $\pi/4, 3\pi/4$ depending on whether we are moving forward or backward (+/-ve x direction). This ensures that the cycles all grow upwards. Since we have the freedom to place the cycles where we want, we space them vertically on the matching so that they are very far apart (non-intersecting). Also, if the special edge of each cycle was between adjacent vertical lines then this edge would not pass through any other vertex of the graph either. Then, the only thing we would need is that the final edge in each cycle is drawn with the same slope. Figure 3 illustrates this and the next remark is followed by a formal description of the problem.

Remark 2.7 *In [E05] a similar strategy of drawing the matching on vertical lines was employed. However, the cycles were drawn with alternating $\pi/4, 3\pi/4$ slopes for adjacent edges, so that the cycles were not "growing upwards" as in our construction. It leads to a different algebraic formulation of the problem giving tight bounds for the case when the cubic graph contains a Hamiltonian cycle.*

Let M be a matching in G. Each cycle C in $E(G) \setminus M$ can be represented as a cyclic sequence $C = (v_1, \ldots, v_k)$, where each v_i is an element of M. The sequence represents the elements of M as we go around the cycle. We can assume (by Claim 2.4) that $k \geq 4$. An *edge* of C by definition is (v_i, v_{i+1}) (all indices are understood mod k), which is although formally a pair formed by two distinct elements of M, also corresponds to an actual edge of the cycle. Notice that each element of M is either shared by two cycles or occurs twice in a single cycle.

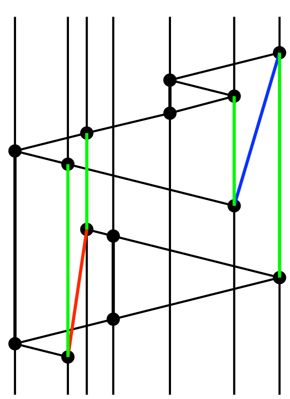

Figure 4: Distinguished 'points' of Figure 1 marked in green

We now want to pick a distinguished edge (as in Figure 4) (v_i, v_{i+1}) in C (and in other cycles) such that the set of distinguished edges will satisfy certain properties.

Notation: Points of the collection of distinguished edges from all cycles form the set of *distinguished points*. We would hope that distinguished points corresponding to a distinguished edge can be drawn as adjacent vertical lines for all cycles so that this would naturally enforce that the distinguished edge would not go through any other vertex of the graph. (Recall that the word "point" somewhat counter-intuitively refers to the elements of M. Also, in our geometric view these are vertical lines)

Definition 2.8 *Two cycles are* connected *if they share a distinguished point, and two cycles* belong to the same component *if they can be reached one from another by going through connected cycles. In other words, we define a graph on the cycles that we call the* **cycle-connectivity graph.** *Notice that in this graph each cycle can have at most two neighbors, thus the graph is a union of paths and cycles. The set of distinguished points associated with the component, where cycle C belongs is denoted by $D(C)$. (Clearly, if C_1 and C_2 belong to the same component, then $D(C_1) = D(C_2)$).*

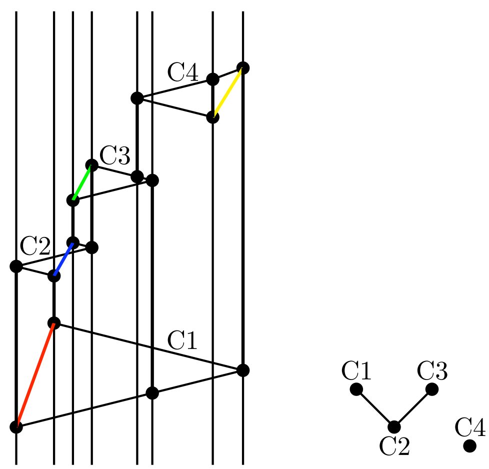

Figure 5: Graph and its connectivity graph

Remark 2.9 *Notice that in the cycle-connectivity graph two cycles are not necessarily connected if they share a point but only if they share a distinguished point. We can define another graph, where two cycles are connected if they are connected through any point. It is easy to see that G is connected iff the latter graph is connected.*

Condition I: The cycle-connectivity graph does not contain cycles (only paths). Equivalently, we can enumerate the distinguished points associated with the cycles of a component in some linear order $y_1, \ldots, y_l$ in such a way that the pairs of consecutive points of this order are exactly the distinguished edges associated with the cycles in the component.

Condition II: In each component there is at most one cycle C such that $C \subseteq D(C)$.

Assume that the lines of the matching are ordered $v_1, ..., v_n$. From Condition I we can ensure that every distinguished edge takes up two adjacent lines in this ordering. A drawing of these lines would be completely determined by the distance between consecutive lines. If v_i, v_{i+1} form a distinguished edge of the k^{th} cycle, then call the distance between these lines x_k. Otherwise fix this distance to be some arbitrary positive constant c_i. This is illustrated in Figure 6.

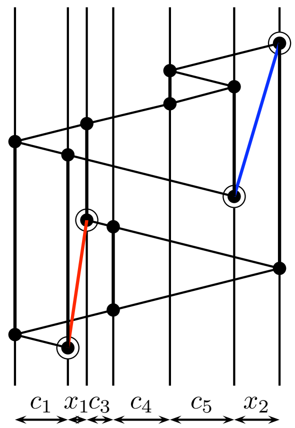

Figure 6: Definition of variables x_i and c_i

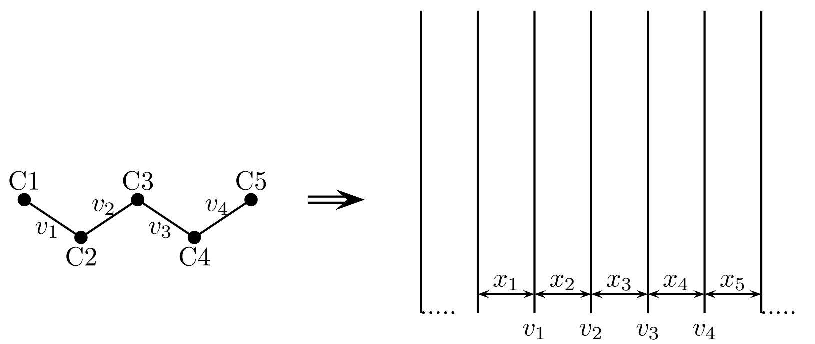

Figure 7: Paths of cycles will have adjacent distinguished edges in the drawing(because of the distinguished point they share). Hence it is necessary to not have cycles in the connectivity graph.

Now draw a cycle by starting at one of the distinguished points and first drawing the path obtained by removing the distinguished edge. If an edge of the cycle is v_k, v_l where $k < l$ then use a slope of $\pi/4$ and $3\pi/4$ otherwise. Notice that the vertical distance travelled across this edge is equal to the distance between the lines v_k and v_l. Hence the slope of the distinguished edge would look like $g_i = \frac{\mathcal{L}_i(\mathbf{x})}{x_i}$ where $\mathcal{L}_i(\mathbf{x}) = a_{i,0} + \sum_{j=1}^{n} a_{i,j}x_j$ for $1 \leq i \leq m$ (m being the number of cycles) is a linear equation on $\mathbf{x}$ with non-negetive integer coefficients. We will use the following solvability theorem to ensure that these slopes can always be matched.

Theorem 2.10 *Let $\mathcal{L}_i(\mathbf{x}) = a_{i,0} + \sum_{j=1}^{n} a_{i,j}x_j$ for $1 \leq i \leq n$ be linear forms, such that all coefficients are non-negative. Define a directed graph, $\mathcal{G} = \mathcal{G}(\overline{\mathcal{L}})$ with vertex set $V(\mathcal{G}) = \{0, 1, \ldots, n\}$ and edge set $E(\mathcal{G}) = \{(j, i) \mid a_{i,j} \neq 0\}$. Let $g_i = \frac{\mathcal{L}_i(\mathbf{x})}{x_i}$ for $1 \leq i \leq n$. Assume that in $\mathcal{G}(\overline{\mathcal{L}})$ every node can be reached from 0. Then*

$$g_1(\mathbf{x}) = g_2(\mathbf{x}) = \cdots = g_n(\mathbf{x}) \tag{1}$$

has an all-positive solution.

This theorem is proved in section 3.

Definition 2.11 *We define $r(i) = dist(0, i)$ in the above graph $\mathcal{G}(\overline{\mathcal{L}})$ and for a cycle C if the variable was x_i for its distinguished edge, we would denote $r(C)$ to mean $r(i)$.*

Theorem 2.12 *If Conditions I and II hold then we can use Theorem 2.10 to prove that every connected graph G is implementable with four directions.*

Proof. Condition I ensures that the slope associated with the distiguished edge of each cycle i can be expressed as $g_i(\mathbf{x})$ (as we have seen). Condition II is sufficient for the reachability condition (for $\mathcal{G}$) of Theorem 2.10. We will in fact show that $r(C) \leq 2$ for every cycle C. The linear expression for cycle C has a non-zero constant term iff $C \setminus D(C) \neq \emptyset$. Consider a fixed component. By Condition II all cycles, except perhaps one, have associated linear expressions with non-zero contant terms, therefore they have $r = 1$.

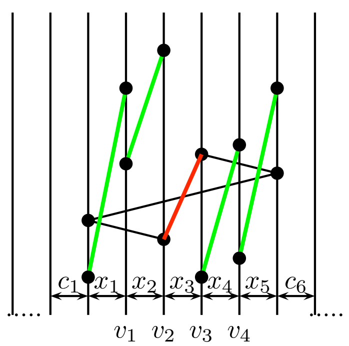

Figure 8: $r(C) \neq 1$ if all edges of the cycle span over adjacent distingished edges. But all inbetween v_i's in the figure have both points used up by cycles. So C could atbest be a 4 cycle (since the graph is triangle free) using up the first and the last vertical lines of this contiguous block and one distinguished edge. The green edges represent the distinguished edges of other cycles

It is sufficient to show that the single cycle C for which $C \subseteq D(C)$, if exists, has $r(C) = 2$. Indeed, let $y_1, \ldots, y_l$ be the distinguished points belonging to this component in this linear order, and let y_p and y_{p+1} be the distiguished points that belong to cycle C. Since C is at least a four cycle, it either contains some other $y_{p'} \notin \{y_p, y_{p+1}\}$, in which case indeed, it is geometrically easy to see that one of the other variables from the component has to occur in $\mathcal{L}_C$ or C is a four cycle and both y_p and y_{p+1} occur with multiplicity two in it. In the latter case C would form a separate K_4 component, thus $G = K_4$. In the former case the variable has $r = 1$, so $r(C) = 2$. □

We are left with proving that we can pick distinguished edges from the cycles such that Conditions I and II are satisfied. Indeed, start from any cycle, and pick an edge for a distinguished edge, which has at least one endpoint y that is common with a different cycle. If there is none, the cycle is the single (Hamiltonian) cycle, and if we distinguish any edge, Conditions I and II are clearly satisfied. Otherwise, in the cycle that contains y, pick one of the two edges not covering

y, look at the other end-point, y', of this edge, look for another cycle that contains y', etc. The process ends when we get back to any cycle (including the current one) that has already been visited. There is one reason for back-track and this is when we return to the other end-point, z, of the starting edge. In this case we choose the other edge (recall we always have two choices). It would be fatal to get back to z, since then Condition I would not hold.

Assume that the above procedure has gone through. Then we have distinguished at most three points from every cycle. But this is not all. We have to do the same procedure from z as well. The procedure terminates when we encounter a cycle that has already been encountered. Thus in the final step we might create a fourth distinguished node in one of the cycles, but only in one of them. This can be the single cycle C in the component for which $C \subseteq D(C)$. And because the graph is triangle-free, all the other components would have $C \setminus D(C) \neq \phi$.

Once we are done with creating the first component, we select a cycle not involved in it, and start the same procedure as before with the only difference that in subsequent rounds we also stop if we encounter a cycle visited in one of the previous rounds. It is easy to see, that now for the distinguished edges that we have selected Conditions I and II hold.

3 Solvability

Before we prove Theorem 2.10, we will look at the following special case when all the constant terms in $\mathcal{L}_i$ are positive.

Theorem 3.1 *Let $B_1, \ldots, B_n > 0$ be positive constants, $\mathcal{L}_i(\mathbf{x}) = \sum_{j=1}^n a_{i,j} x_j$ for $1 \leq i \leq n$ be linear forms. Let $g_i = \frac{B_i + \mathcal{L}_i(\mathbf{x})}{x_i}$ for $1 \leq i \leq n$. Then*

$$g_1(\mathbf{x}) = g_2(\mathbf{x}) = \cdots = g_n(\mathbf{x}) \tag{2}$$

has an all-positive solution.

Proof. The intuition behind the proof is this: Let ϵ be very small and $\alpha_1, \ldots, \alpha_n > 0$ be fixed. If we set $x_i = \epsilon B_i \alpha_i^{-1}$ then $g_i(\mathbf{x}) \approx \epsilon^{-1} \alpha_i$. In particular, let $\overline{\alpha}$ range in the $[1, 2]^n$ solid cube. Then, if ϵ is small enough, the vector $(g_1(\mathbf{x}), \ldots, g_n(\mathbf{x}))$ will range roughly in the $[\epsilon^{-1}, 2\epsilon^{-1}]^n$ cube, thus $\epsilon^{-1}(1.5, \ldots, 1.5)$, which is the center of this cube, has to be in the image.

To make this proof idea precise we will use the following version of Brower's well known fix point theorem:

Theorem 3.2 (Brower) *Let $f : [1, 2]^n \to [1, 2]^n$ be a continuous function. Then f has a fix point, i.e. an $\mathbf{x}_0 \in [1, 2]^n$ for which $f(\mathbf{x}_0) = \mathbf{x}_0$.*

We will use the fix point theorem as below. We first define

$$h(\alpha_1, \ldots, \alpha_n) = (\epsilon g_1(\mathbf{x}), \ldots, \epsilon g_n(\mathbf{x})),$$

where $\mathbf{x} = \epsilon(\alpha_1^{-1} B_1, \ldots, \alpha_n^{-1} B_n) = \epsilon \mathbf{x}'$, and we think of ϵ as some fixed positive number. Notice that $\mathbf{x}'$ is just a function of $\overline{\alpha}$, independent of ϵ. It is sufficient to show that if ϵ is small enough, there are $\alpha_1, \ldots, \alpha_n$ such that $h(\overline{\alpha}) = (1.5, \ldots, 1.5)$, since then $\mathbf{x}$ satisfies (2) with common value $1.5\epsilon^{-1}$. We have:

$$\epsilon g_i(\mathbf{x}) = \epsilon \frac{B_i + \mathcal{L}_i(\mathbf{x})}{\epsilon \alpha^{-1} B_i} = \alpha_i (1 + \epsilon B_i^{-1} \mathcal{L}_i(\mathbf{x}')).$$

Here we used that $\mathcal{L}_i(\epsilon\mathbf{x}') = \epsilon\mathcal{L}_i(\mathbf{x}')$. We would like to have

$$\alpha_i(1 + \epsilon B_i^{-1}\mathcal{L}_i(\mathbf{x}')) = 1.5 \quad \text{for } 1 \le i \le n. \tag{3}$$

Define

$$\begin{aligned} K &= \max_i \sup_{\overline{\alpha}\in[1,2]^n} B_i^{-1}\mathcal{L}_i(\mathbf{x}'); \\ \epsilon &= 1/(10K). \end{aligned}$$

To use the fix point theore we consider the map

$$f : (\alpha_1, \ldots, \alpha_n) \to \left(\frac{1.5}{1 + \epsilon B_1^{-1}\mathcal{L}_1(\mathbf{x}')}, \ldots, \frac{1.5}{1 + \epsilon B_n^{-1}\mathcal{L}_n(\mathbf{x}')}\right)$$

on the cube $[1, 2]^n$. The image is contained in $[1, 2]^n$, since if $\overline{\alpha} \in [1, 2]^n$ then for $1 \le i \le n$ we have

$$1 < \frac{1.5}{1 + 0.1} = \frac{1.5}{1 + \epsilon K} \le \frac{1.5}{1 + \epsilon B_i^{-1}\mathcal{L}_i(\mathbf{x}')} \le \frac{1.5}{1 - \epsilon K} = \frac{1.5}{1 - 0.1} < 2.$$

Therefore, by Theorem 3.2 there is an $\overline{\alpha} \in [0, 1]^n$ such that $\alpha_i = \frac{1.5}{1+\epsilon B_i^{-1}\mathcal{L}_i(\mathbf{x}')}$ for $1 \le i \le n$, which is equivalent to (3). □

In Theorem 3.1 all linear forms have non-zero contant terms. We can, however generalize this to Theorem 2.10. We discuss its proof below.

Remark 3.3 *The non-negativity of the coefficients can be relaxed such that the theorem becomes a true generalization of Theorem 3.1, Since the more general condition is slightly technical, we will stay with the simpler non-negativity condition, which is sufficient for us.*

Proof. For $1 \le i \le n$ let $r(i) = dist(0, i)$ in $\mathcal{G}(\overline{\mathcal{L}})$. (In Theorem 3.1 all $r(i)$ was 1.) Define

$$x_i = \epsilon^{r(i)} x_i',$$

where $\epsilon > 0$ will be a small enough number that we will approriately fix later, but as of now we think about it as a quantity tending to zero. We can rewrite (2) as:

$$\epsilon g_1(\mathbf{x}) = \epsilon g_2(\mathbf{x}) = \cdots = \epsilon g_n(\mathbf{x}).$$

If we fix $\mathbf{x}'$ and tend to zero with epsilon, then

$$\epsilon g_i(\epsilon\mathbf{x}') \to \frac{\beta_i(\mathbf{x}')}{x_i'},$$

where $\beta_i(\mathbf{x}') = a_{i,0}/x_i'$ if $r(i) = 1$, otherwise

$$\beta_i = \sum_{j:\ r(j)=r(i)-1} a_{i,j} x_j'.$$

We can now solve the system

$$\frac{\beta_i(\mathbf{x}')}{x_i'} = 1.5$$

and even the system

$$\frac{\beta_i(\mathbf{x}')}{x'_i} = \alpha_i, \tag{4}$$

where $1 \le \alpha_i \le 2$ for $1 \le i \le n$, and arbitrary otherwise. Indeed, the solution can be obtained iteratively, by first computing the values of the variables x_i with $r(i) = 0$, then with $r(i) = 1$, etc. We can again use the fix point theorem of Brower to show that if ϵ is sufficiently small, the system

$$\epsilon g_i(\epsilon \mathbf{x}') = 1.5 \qquad \text{for } 1 \le i \le n$$

has a solution. For this we again parameterize $\mathbf{x}'$ with $\overline{\alpha}$. When $\overline{\alpha}$ ranges in the solid cube $[1,2]^n$ then $\mathbf{x}'$ will range in some domaim D, where we obtain D by by solving the system (4) for all $\alpha \in [1,2]^n$. Now we have to set ϵ small enough such that everywhere in D it should hold that

$$0.9 \le \frac{\beta_i(\mathbf{x}')/x'_i}{\epsilon g_i(\epsilon \mathbf{x}')} = \frac{\alpha_i}{\epsilon g_i(\epsilon \mathbf{x}')} \le 1.1 \qquad \text{for } 1 \le i \le n. \tag{5}$$

This is easily seen to be possible, since D is contained in a closed cube in the strictly positive orthant. We then apply the fix point theorem to

$$f : \overline{\alpha} \to \overline{\gamma},$$

where

$$\gamma_i = \frac{1.5\alpha_i}{\epsilon g_i(\epsilon \mathbf{x}')}.$$

The fix point theorem applies, since the range of f remains in the $[0.9 \cdot 1.5, 1.1 \cdot 1.5]^n \subset [1,2]^n$ cube by Equation (5). For the fixed point $\alpha_i = \frac{1.5\alpha_i}{\epsilon g_i(\epsilon \mathbf{x}')}$ for $1 \le i \le n$, which implies $\epsilon g_i(\epsilon \mathbf{x}') = 1.5$ for $1 \le i \le n$. □

4 Open Problems

For disconnected cubic graphs in which all components are bridgleless and triangle-free, we can simultaneously solve for the slope equations in all components and hence draw the graph with four slopes. If some of the components have a bridge then we use Theorem 2.1 to draw these components. Hence, in general, disconnected cubic graphs would require five slopes. This is the same as the bound achieved in [KPPT].

Another natural question that arises by looking at the drawings of K_4 and peterson graph is what cubic graphs can be drawn with only the standard four directions (N,E,NE,NW)? From the results in section 2.1, it is evident that cubic graphs with a cutvertex or a two-vertex disconnecting set or a bridge or two-edge disconnecting set can be drawn with the standard directions. Hence, any graph that requires a non standard slope must be three vertex and edge connected.

Another interesting question arises if we constrain that any two vertices that are not connected by an edge in the graph must have a different slope between them than the slopes used in the graph. This is a strengthening of requiring specific slopes for edges of the graph. In particular, such a drawing of the graph avoids colinearity of points (thereby enforcing general position of the points). It was shown in [KPP07] that seven slopes suffice for cubic graphs. Here again they use six slopes for subcubic graphs and the additional edge requires one more slope. So a natural question would be if six was sufficient.

Another related open problem is to find bounds for the slope number of degree-4 graphs. It is believed that this is unbounded.

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